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Mechanics 3

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Abstract

The submited textbook is prepared for students of Faculty of mechanical engineering and Faculty of applied sciences at the University of West Bohemia in Pilsen. It can serve as background for studying objects ĂMechanics 3ò, ĂMathematical theory of vibrationò and partly for ĂComputational methods in mechanicsò. It is supossed that the reader has knowledge from dynamics of mass point, rigid body and basic kinematics. This textbook contains some special parts from rotation and spherical motion for which dual description by Eulerôs and Cardanôs angles is used. Chapter Analytical mechanics brings some original ideas esspecially virtual work principle for systems having higher number of generalized coordinates then number of degrees of freedom whose application results are compared with those ones obtained by mixed Lagrangeôs equations. Strictly matrix notation is used in whole textbook and for this reason the skills of matrix calculus is required.

Contents

1. Dynamics of rigid body rotation

The rotating rigid body is depicted in fig. 1.1. The stationary coordinate system is marked by *x, y, z* while moving coordinate system connected with body is marked by ξ, η, ζ .

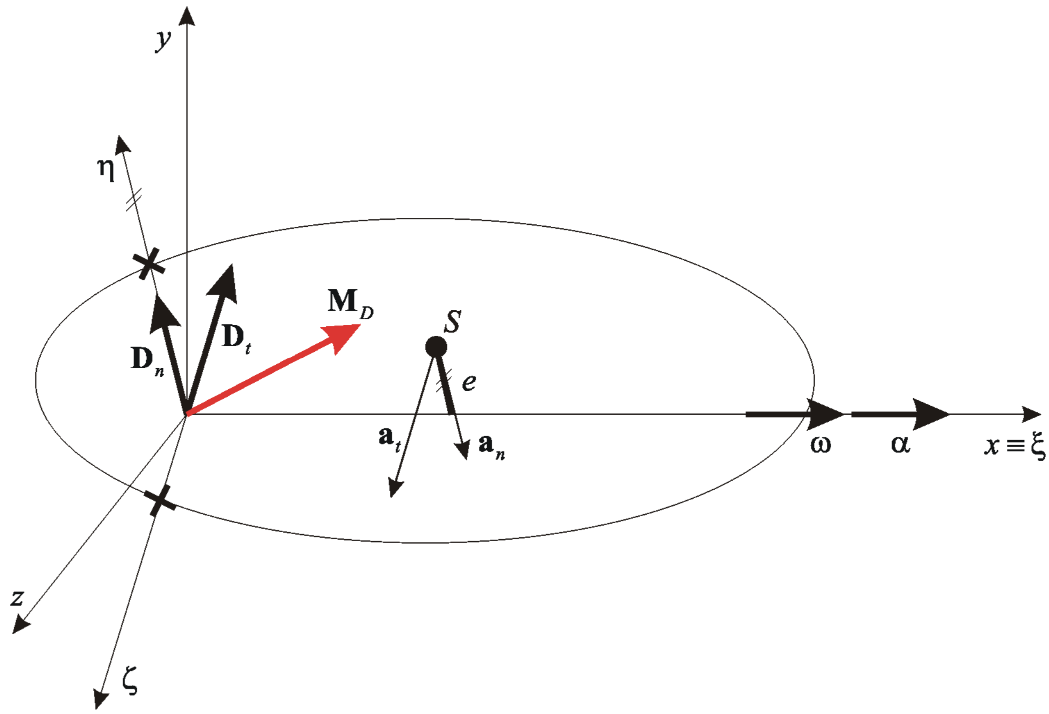

Fig. 1.1

The axis ξ of the moving (rotating) system is identical with the axis *x* of the stationary system. The vector of angular speed $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ are collinear with ξ and *x*. Let us introduce the constant inertia matrix of the body with respect of the moving coordinate system ξ, η, ζ having the form

$$\mathbf{J}_{\xi\eta\zeta} = \begin{bmatrix} I_{\xi}, & -D_{\xi\eta}, & -D_{\xi\zeta} \\ -D_{\xi\eta}, & I_{\eta}, & -D_{\eta\zeta} \\ -D_{\xi\zeta}, & -D_{\eta\zeta}, & I_{\zeta} \end{bmatrix}. \tag{1.1}$$

The individual components of inertia matrix are moments and products of inertia with respect to coordinate system ξ, η, ζ . Body moment of momentum can be expressed in form

$$\mathbf{L} = \mathbf{J}\boldsymbol{\omega} \ , \tag{1.2}$$

where $\boldsymbol{\omega} = [\omega, 0, 0]^T$ is vector of result angular speed. Let us remind that the quantities in (1.2) are expressed in the coordinate system ξ, η, ζ in spite of the fact that the vector $\boldsymbol{\omega}$ has the same coordinates in both systems. The body momentum can be expressed as

$$\mathbf{H} = m\mathbf{v}_S = m\boldsymbol{\omega} \times \mathbf{r}_S, \tag{1.3}$$

where $\mathbf{v}_S$ is velocity vector of the centre of gravity. From the Fig. 1.1 we can see the varying direction of the momentum vector. Kinetic energy of the rotating body can be written according to the well known relation

$$E_k = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{L}, \tag{1.4}$$

which respecting (1.2) can be rewritten in form

$$E_k = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{J}\boldsymbol{\omega}\,. \tag{1.5}$$

Inertia effects on the rotating body are composed of $\mathbf{D}_n$ and $\mathbf{D}_t$ and the moment of inertia forces $\mathbf{M}_D$ see Fig. 1.1. To obtain these quantities let us remind the important relations of basic dynamics of mass point system. The first expresses change of momentum in the integral and differential form, respectively

$$\mathbf{H} - \mathbf{H}_0 = \int_0^t \mathbf{F}dt \quad \text{resp.} \quad \dot{\mathbf{H}} = \mathbf{F}, \tag{1.6}$$

where $\mathbf{H}$ is total momentum of the mass point system (mass body in this case) and $\mathbf{F}$ is resultant of the external forces treating with the body. The second relation corresponds to the change of moment of momentum in the integral and differential form, respectively

$$\mathbf{L} - \mathbf{L}_0 = \int_0^t \mathbf{M}dt\,, \quad \text{resp.} \quad \dot{\mathbf{L}} = \mathbf{M}, \tag{1.7}$$

where $\mathbf{L}$ is resultant moment of momentum of the mass point system (mass body in this case) and $\mathbf{M}$ is the resultant moment of external forces and moments. (To transfer external forces to one point we have to add corresponding moments). Following d'Alembertős principle we can write the condition of external and inertia effect balance which can be expressed by two relations

$$\mathbf{F} + \mathbf{D} = \mathbf{0}, \qquad \mathbf{M} + \mathbf{M}_D = \mathbf{0}, \tag{1.8}$$

where $\mathbf{D}$ is resultant inertia force and $\mathbf{M}_D$ is resultant moment of inertia forces. Respecting (1.8) we can rewrite the second relations in (1.6) and (1.7) into the forms

$$\mathbf{D} = -\dot{\mathbf{H}}, \qquad \mathbf{M}_D = -\dot{\mathbf{L}}\,. \tag{1.9}$$

These quantities are understood as replacement in the centre of gravity or in fixed point (e.g. in the origin of coordinate system). The resultant inertia force we can obtain according to (1.9) and respecting (1.3) in form (mass is constant)

$$\mathbf{D} = -m\dot{\mathbf{v}}_S = -m(\mathbf{a}_n + \mathbf{a}_t) = \mathbf{D}_n + \mathbf{D}_t, \tag{1.10}$$

where

$$a_n = |\mathbf{a}_n| = e\omega^2, \quad \omega = |\boldsymbol{\omega}|, \tag{1.11}$$

and

$$a_t = |\mathbf{a}_t| = e\alpha, \quad \alpha = |\dot{\boldsymbol{\omega}}|$$

are normal and tangential accelerations of centre of gravity, respectively and

$$\boxed{\mathbf{D}_n = -m\,\mathbf{a}_n, \qquad \text{resp.} \qquad \mathbf{D}_t = -m\,\mathbf{a}_t} \tag{1.12}$$

are corresponding normal and tangential inertia force, respectively. Let us remind again that both inertia forces are acting in the origin of the coordinate system. Directions of these inertia forces are depicted in Fig. 1.1. Let us turn back to the second relation in (1.9) which serves to the expression of moment of inertia forces. According to this relation the vector of resultant inertia forces can be obtained from the differentiation of moment of momentum with respect to time. The differentiation has to be performed in the stationary coordinate system. Respecting fact that the moment of momentum (1.2) is expressed in the moving coordinate system connected with rotating body we can rewrite the second relation in (1.9) into form

$$\boxed{\mathbf{M}_D = -\dot{\mathbf{L}} = -\dot{\mathbf{L}}\big|_{\xi\eta\zeta} - \boldsymbol{\omega} \times \mathbf{L}}, \tag{1.13}$$

where $-\dot{\mathbf{L}}\big|_{\xi\eta\zeta}$ means the derivation of moment of momentum expressed in rotating coordinate system $\xi\eta\zeta$ with respect to time. This relation we can explain in the simplest way: The vector **a** expressed both in stationary coordinate system xyz generated by the base $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and in the moving coordinate system $\xi\eta\zeta$ generated by the base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3. \tag{1.14}$$

Differentiating both sides of (1.14) with respect to time and respecting that the base $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is constant we can come to

$$\dot{a}_x\mathbf{i} + \dot{a}_y\mathbf{j} + \dot{a}_z\mathbf{k} = \dot{a}_1\mathbf{e}_1 + a_1\dot{\mathbf{e}}_1 + \dot{a}_2\mathbf{e}_2 + a_2\dot{\mathbf{e}}_2 + \dot{a}_3\mathbf{e}_3 + a_3\dot{\mathbf{e}}_3. \tag{1.15}$$

Taking into account the constant magnitude of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, their differentiations with respect to time can be expressed in the simple way as

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i, \quad i = 1, 2, 3. \tag{1.16}$$

After substitution of the last relations into (1.15) it follows

$$\dot{a}_x\mathbf{i} + \dot{a}_y\mathbf{j} + \dot{a}_z\mathbf{k} = \dot{a}_1\mathbf{e}_1 + \dot{a}_2\mathbf{e}_2 + \dot{a}_3\mathbf{e}_3 + \boldsymbol{\omega} \times \underbrace{\left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3\right)}_{\mathbf{a}}. \tag{1.17}$$

The Eq. (1.17) can be simply rewritten into matrix form

$$\dot{\mathbf{a}} = \dot{\mathbf{a}}\big|_{\xi\eta\zeta} + \boldsymbol{\omega} \times \mathbf{a}, \tag{1.18}$$

which proves a validity of (1.13). Substitution (1.2) into (1.13) we can come to

$$\mathbf{L} = \begin{bmatrix} I_\xi\omega \\ -D_{\xi\eta}\omega \\ -D_{\xi\zeta}\omega \end{bmatrix}, \qquad \mathbf{M}_D = \begin{bmatrix} -I_\xi\alpha \\ D_{\xi\eta}\alpha \\ D_{\xi\zeta}\alpha \end{bmatrix} - \begin{bmatrix} 0 \\ D_{\xi\zeta}\omega^2 \\ -D_{\xi\eta}\omega^2 \end{bmatrix} = \begin{bmatrix} -I_\xi\alpha \\ D_{\xi\eta}\alpha - D_{\xi\zeta}\omega^2 \\ D_{\xi\zeta}\alpha + D_{\xi\eta}\omega^2 \end{bmatrix}. \tag{1.19}$$

Examples on rotating motion

Example 1.1

The ring of the radius r having mass m is depicted in Fig. 1.2. This ring is linked with foundation by four elastic felloes clamped at the ends to the foundation and to the ring, respectively. Determine the eigenfrequency of the ring torsional vibration. Let us suppose small displacements, massless felloes and ring thickness.

Inputs: m, r, E-Young modulus of felloes, J-moment of inertia of the felloe cross section with respect to the central axis of the cross section perpendicular to the picture plane

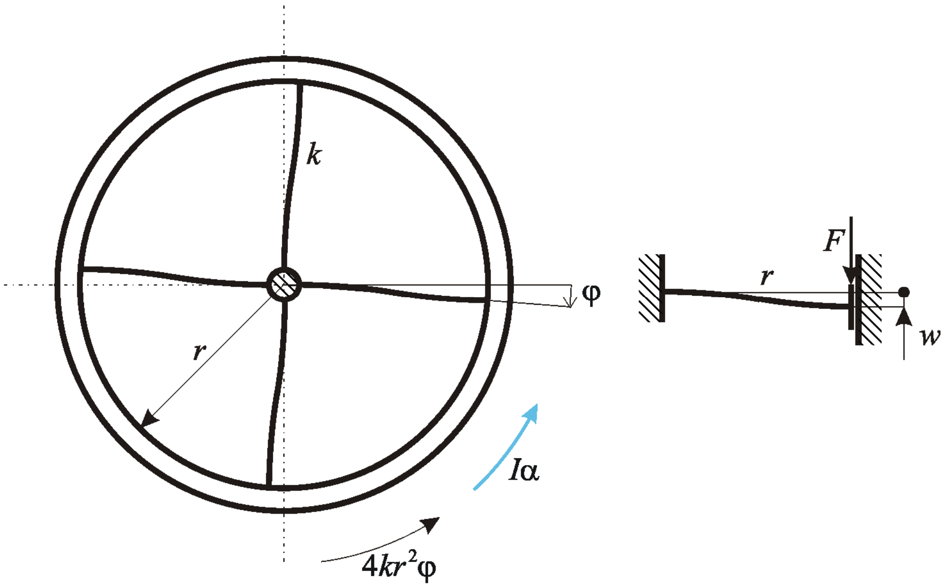

Fig. 1.2

Solution:

For the reason that ring rotates about central axis of inertia the inertial forces $\mathbf{D}_n$ a $\mathbf{D}_t$ are equal to zero $(e = 0)$. The moment of inertia forces has only one nonzero component in the direction of rotation axis $-I_\xi \alpha$. For brevity we leave out the subscript ξ. As a starting point we determine stiffness k of one felloe. Let us load the beam with zero slopes at both ends according to Fig. 1.2 and get the relation for displacement and corresponding stiffness in the form

$$w = \frac{Fr^3}{12EJ} \Rightarrow k = \frac{12EJ}{r^3}. \tag{1.20}$$

Respecting assumption of small displacements we can express the bend at the end of beam in form $w = r\varphi$. The moment of reversible forces can be written as

$$M_V = 4kwr = 4kr^2\varphi. \tag{1.21}$$

Let us substitute (1.20) into the last relation and have

$$M_V = \frac{48EJ}{r}\varphi. \tag{1.22}$$

Neglecting the ring thickness we can express come to the moment of inertia in form

$$I = mr^2. \tag{1.23}$$

From the moment balance condition and from the relation $\alpha = \ddot{\varphi}$ we can obtain the equation of motion describing torsion vibration of the ring

$$mr^2\ddot{\varphi} + \frac{48EJ}{r}\varphi = 0. \tag{1.24}$$

Dividing the last equation by mr^2 we can come to

$$\ddot{\varphi}+\underbrace{\frac{48EJ}{mr^3}}_{\Omega^2}\varphi=0, \tag{1.25}$$

from which follows the ring torsion eigenfrequency in form

$$\Omega=\sqrt{\frac{48EJ}{mr^3}}. \tag{1.26}$$

Example 1.2

The bar 3 rotating by constant angle speed ω_0 about ξ axis is depicted in Fig. 1.3. One end of the bar is connected to the element 2 by spherical coupling. Another end is linked with base frame by massless fibre of the length b. Let us determine the slope angle φ. In case the angle achieves its maximal value $\varphi_{\max}$ let us determine the force acting in the fibre.

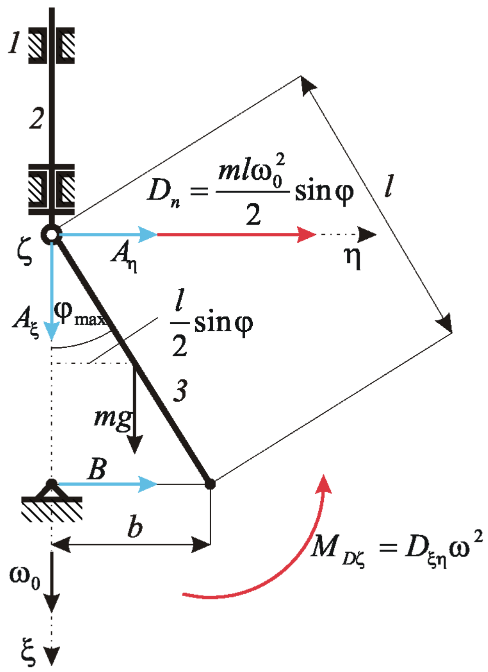

Fig. 1.3

Solution:

Only one of the inertia forces (1.12) is nonzero

$$D_n=\frac{ml\omega_0^2}{2}\sin\varphi. \tag{1.27}$$

Simultaneously only one component of the moment of inertia forces (1.19) is nonzero

$$M_{D\zeta}=D_{\xi\eta}\omega_0^2. \tag{1.28}$$

These both inertia effects are marked by red colour in Fig. 1.3. To obtain the inertia force moment (1.28) we have to determine product of inertia $D_{\xi\eta}$. From Fig. 1.4 we can come to the relation (the bar is prismatic and homogeneous)

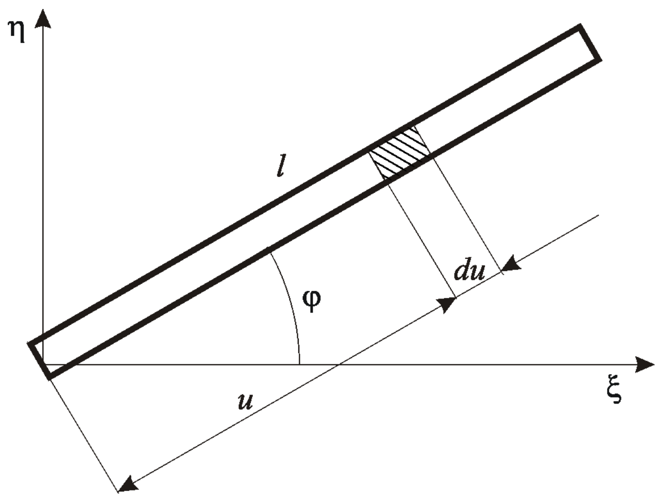

Fig. 1.4

$$D_{\xi\eta} = \int \xi\eta dm = \int_0^l \frac{m}{l} u^2 \sin\varphi \cos\varphi \, du = \frac{ml^2}{3} \sin\varphi \cos\varphi. \tag{1.29}$$

Firstly we will suppose that the angle slope lies in the region

$$\varphi \in (0, \varphi_{\max}), \qquad \varphi_{\max} = \arcsin\frac{b}{l}. \tag{1.30}$$

The conditions of the dynamic balance have form

$$\begin{aligned} &A_\xi + mg = 0, \\ &\underbrace{B}_{0} + A_\eta + \frac{ml\omega_0^2}{2}\sin\varphi = 0, \\ &\underbrace{\frac{ml^2}{3}\sin\varphi\cos\varphi}_{D_{\xi\eta}}\omega_0^2 - mg\frac{l}{2}\sin\varphi + \underbrace{B}_{0}\sqrt{l^2 - b^2} = 0. \end{aligned} \tag{1.31}$$

It follows from the equation (1.31) that the vertical coupling force remains constant having value $A_\xi = -mg$. The coupling force A_η can be obtained from the second equation (1.31) but this force depends on the angle slope φ. For this reason it is necessary at the earliest to solve the third equation (1.31) which has after small arrangement form

$$\sin\varphi\left(\frac{l\omega_0^2}{3}\cos\varphi - \frac{g}{2}\right) = 0. \tag{1.32}$$

This equation can have (but need not) these solutions:

a) $\sin\varphi = 0 \Rightarrow \varphi = 0, \pi$, (1.33)

where the solution $\sin\varphi = \pi$ corresponds to the unstable position and moreover does not lie in the required region given by the fibre length. We will not take into account this solution. The next solution has form

b) $\dfrac{l\omega_0^2}{3}\cos\varphi - \dfrac{g}{2} = 0 \Rightarrow \cos\varphi = \dfrac{3g}{2l\omega_0^2}$, (1.34) but this solution need not exist in case the value $\dfrac{3g}{2l\omega_0^2}$ lies out of the region $\langle -1, 1\rangle$ (the fraction is positive and for this reason the region is reduced to $(0,1\rangle$). If this value lies in region $(0,1\rangle$ we can write

$$\varphi = \pm \arccos\left(\frac{3g}{2l\omega_0^2}\right). \tag{1.35}$$

The interesting conclusion follows from the relation (1.34). In case the angle speed is small but nonzero the angle slope stays zero until the critical value of the angle speed ($\dfrac{3g}{2l\omega_0^2} = 1 = \cos\varphi.$)

$$\omega_{0,krit} = \pm\sqrt{\frac{3g}{2l}}. \tag{1.36}$$

In case the angle slope would be larger than $\varphi_{\max} = \arcsin\dfrac{b}{l}$, it is necessary to respect nonzero value of the reaction force in fibre B and take into account the angle slope value equal to $\varphi_{\max}$. Solution of the set of equations (1.31) in this case contains coupling forces A_ξ, A_η a B.

2. Dynamics of the body spherical motion

The spherical motion is defined in such a way, that one point of the body is fixed. Let us focus on this problem from the kinematic point of view. Rigid body has 6 degrees of freedom (DOF) in 3D space. It means 3 independent displacements and 3 rotations. If one point of the body is fixed 3 displacements are eliminated and 3 rotations remain. These rotations can be described by various manners. One of most used is description by means of Euler's angles [1].

Application of Euler's angles

These angles describing spherical motion of body are depicted in Fig. 2.1

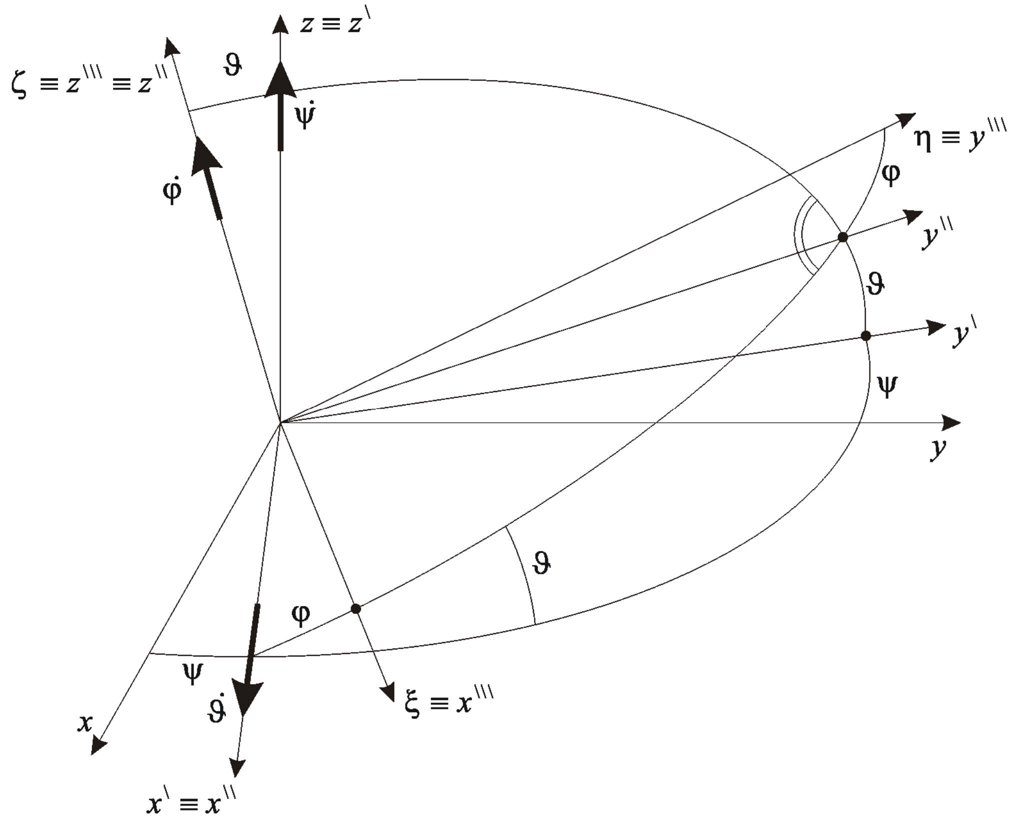

Fig. 2.1

Let us assume that the body is connected with movable coordinate system $\xi\eta\zeta$, which was originally identical with stationary coordinate system xyz . The movable system achieved the current position by means of three sequential rotations.

a) rotation about z axis -angle ψ (precession angle)
b) rotation about $x' \equiv x''$ axis -angle ϑ (nutation angle)
c) rotation about $\xi \equiv z''' \equiv z''$ axis -angle φ (rotation angle)

Time derivations of the mentioned angles (angle speeds) correspond to the individual rotations whose composition forms the spherical motion. For moment of momentum and kinetic energy expression of the arbitrary form body doing spherical motion it is useful to express these quantities in the coordinate system $\xi\eta\zeta$, because the body inertia matrix is time independent in this system. (Kinetic energy is spatial invariant independent of space in which is expressed). Vector of resulting angle speed expressed in coordinate systems $\xi\eta\zeta$ and xyz , respectively, has form

$$\boldsymbol{\omega}_{\xi\eta\zeta} = \begin{bmatrix} \dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi \\ \dot{\psi}\sin\vartheta\cos\varphi - \dot{\vartheta}\sin\varphi \\ \dot{\varphi} + \dot{\psi}\cos\vartheta \end{bmatrix}, \qquad \text{resp.} \quad \boldsymbol{\omega}_{xyz} = \begin{bmatrix} \dot{\vartheta}\cos\psi + \dot{\varphi}\sin\vartheta\sin\psi \\ \dot{\vartheta}\sin\psi - \dot{\varphi}\sin\vartheta\cos\psi \\ \dot{\psi} + \dot{\varphi}\cos\vartheta \end{bmatrix}. \tag{2.1}$$

Body inertia matrix $\mathbf{J}_{\xi\eta\zeta}$ was already described in the first chapter (relation (1.1)). Moment of momentum can be written in similar form as (1.2)

$$\mathbf{L} = \mathbf{J}_{\xi\eta\zeta}\boldsymbol{\omega}_{\xi\eta\zeta} \tag{2.2}$$

and momentum in similar form to (1.3)

$$\mathbf{H} = m\mathbf{v}_S = m\boldsymbol{\omega}_{\xi\eta\zeta} \times \mathbf{r}_S. \tag{2.3}$$

Now it is important to remind the fact that both last vector quantities are expressed in coordinate system $\xi\eta\zeta$. In the similar way as in the first chapter the kinetic energy can be written down as

$$E_k = \frac{1}{2}\boldsymbol{\omega}^T_{\xi\eta\zeta}\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}^T_{\xi\eta\zeta}\mathbf{J}_{\xi\eta\zeta}\boldsymbol{\omega}_{\xi\eta\zeta}. \tag{2.3}$$

The inertia effects (inertia force and moment of inertia forces) can be determined according to the relations (1.9) and (1.13), respectively

$$\mathbf{D} = -\dot{\mathbf{H}}, \qquad \mathbf{M}_D = -\dot{\mathbf{L}}, \tag{2.4}$$

$$\mathbf{M}_D = -\dot{\mathbf{L}} = -\dot{\mathbf{L}}\Big|_{\xi\eta\zeta} - \boldsymbol{\omega} \times \mathbf{L}. \tag{2.5}$$

These relations are complicated for the bodies of general form. For this reason we focus on rotationally symmetrical bodies and the presented methodology will be a little bit simpler.

Use of modified Euler's angles [2]

Respecting that the body own rotation is going on the axis of body symmetry we can say that moments of inertia are identical

$$I_\xi = I_\eta = I. \tag{2.6}$$

All products of inertia are zeros, too. It means that body inertia matrix is diagonal and has form

$$\mathbf{J}_{\xi\eta\zeta} = \mathbf{J}^{\backslash\backslash} = \begin{bmatrix} I, & 0, & 0 \\ 0, & I, & 0 \\ 0, & 0, & I_o \end{bmatrix}, \tag{2.7}$$

where symbol I_o marks an axial moment of inertia with respect to axis of body symmetry. In this case the coordinate system x'', y'', z'', is sufficient because rotation φ does not change the body inertia matrix. Then the corresponding angles are called by modified Eulerôs angles. These angles are depicted in Fig. 2.2

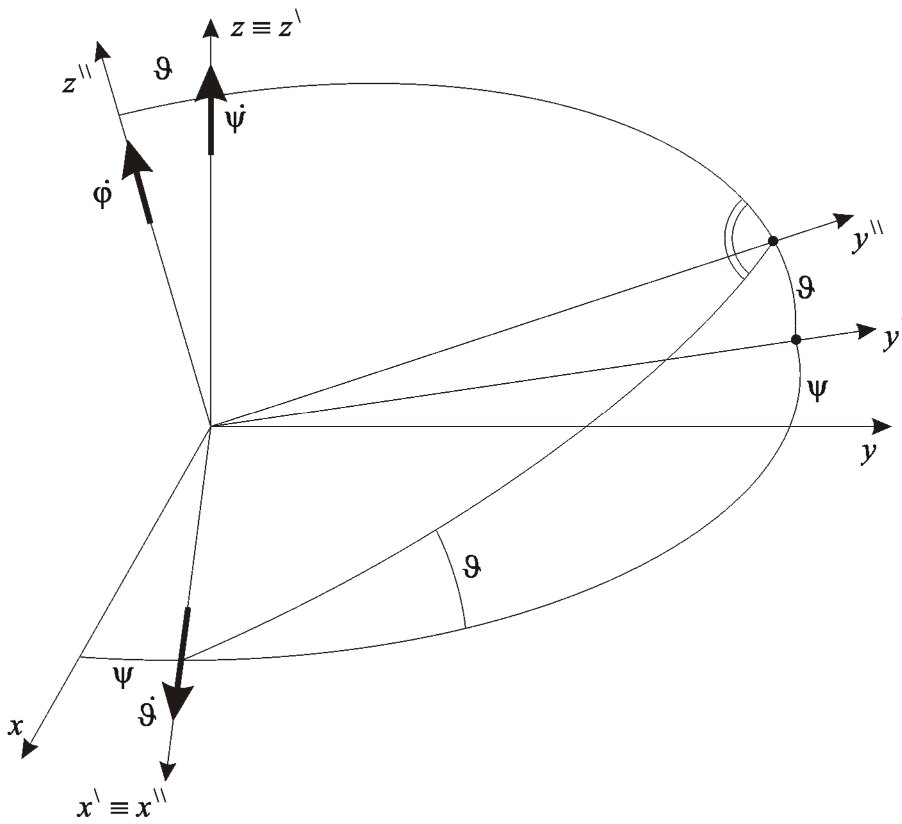

Fig. 2.2

The angle speed vector can be then expressed in ñtwo comma coordinate systemò in form

$$\boldsymbol{\omega}_{\xi\eta\zeta} = \begin{bmatrix} \dot{\vartheta} \\ \dot{\psi}\sin\vartheta \\ \dot{\varphi} + \dot{\psi}\cos\vartheta \end{bmatrix}. \tag{2.8}$$

To express the moment of inertia forces according to (2.5) it is necessary to aware of fact that the motion of ñtwo comma coordinate systemò in relation to base frame is not described by the angle speed vector $\boldsymbol{\omega}_{\xi\eta\zeta}$ but by the vector

$$\boldsymbol{\omega}'' = \begin{bmatrix} \dot{\vartheta} \\ \dot{\psi}\sin\vartheta \\ \dot{\psi}\cos\vartheta \end{bmatrix}. \tag{2.9}$$

The ñtwo comma coordinate systemò does not rotate by the angle speed $\dot{\varphi}$ in relation to the base frame. Moment of momentum vector expressed in ñtwo comma coordinate systemò has form

$$\mathbf{L}'' = \mathbf{J}_{\xi\eta\zeta}\boldsymbol{\omega}_{\xi\eta\zeta} = \mathbf{J}''\boldsymbol{\omega}_{\xi\eta\zeta}. \tag{2.10}$$

To obtain the resulting moment of inertia forces it is necessary to perform the derivation of the moment of momentum according to the relation

$$\mathbf{M}_D = -\dot{\mathbf{L}} = -\dot{\mathbf{L}}^{\backslash\backslash} - \mathbf{q}^{\backslash\backslash} \times \mathbf{L}^{\backslash\backslash}. \tag{2.11}$$

Momentum can be expressed as

$$\mathbf{H} = m\mathbf{v}_S = m\mathbf{q}^{\backslash\backslash} \times \mathbf{r}_S, \tag{2.12}$$

and kinetic energy can be written in form

$$E_k = \frac{1}{2}\boldsymbol{\Upsilon}^T_{\xi\eta\zeta}\mathbf{L}^{\backslash\backslash} = \frac{1}{2}\boldsymbol{\Upsilon}^T_{\xi\eta\zeta}\mathbf{J}_{\xi\eta\zeta}\boldsymbol{\Upsilon}_{\xi\eta\zeta} = \frac{1}{2}\boldsymbol{\Upsilon}^T_{\xi\eta\zeta}\mathbf{J}^{\backslash\backslash}\boldsymbol{\Upsilon}_{\xi\eta\zeta}. \tag{2.13}$$

All vector and tensor quantities (inertia matrix is in fact tensor) are expressed in ñtwo comma coordinate systemò.

Note:

Tensor is mathematical expression of some quantity which subjects to the transformation corresponding to the change of the coordinate system. Zero order tensor is scalar being invariant to coordinate system change. Tensor of the first order is vector subjected to tensor (vector) transformation (vector transformation can be performed by pre-multiplying by transformation matrix). Second order tensor can be expressed as system of quantity coordinates assembled in the form of matrix. The tensor transformation corresponds to pre-multiplying by transformation matrix and post-multiplying by transposed transformation matrix. The transformation of higher order tensors cannot be performed by means of matrix calculus. It is necessary in this case to use a subscript and superscript symbolism [1].

Having performed the indicated operations we can come to

$$\mathbf{L}^{\backslash\backslash} = \mathbf{J}^{\backslash\backslash}\boldsymbol{\Upsilon}_{\xi\eta\zeta} = \begin{bmatrix} I\dot{\vartheta} \\ I\dot{\psi}\sin\vartheta \\ I_o(\dot{\varphi} + \dot{\psi}\cos\vartheta) \end{bmatrix}, \tag{2.14}$$

$$E_k = \frac{1}{2}\left[I\dot{\vartheta}^2 + I\dot{\psi}^2\sin^2\vartheta + I_o(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\right], \tag{2.15}$$

$$\mathbf{M}_D = -\dot{\mathbf{L}}^{\backslash\backslash} - \mathbf{q}^{\backslash\backslash} \times \mathbf{L}^{\backslash\backslash} = \begin{bmatrix} -I\ddot{\vartheta} - \underbrace{I_o\dot{\varphi}\dot{\psi}\sin\vartheta}_{M_{G1}} - (I_o - I)\dot{\psi}^2\sin\vartheta\cos\vartheta \\ -I\ddot{\psi}\sin\vartheta - 2I\dot{\psi}\dot{\vartheta}\cos\vartheta + \underbrace{I_o\dot{\varphi}\dot{\vartheta}}_{M_{G2}} + I_o\dot{\psi}\dot{\vartheta}\cos\vartheta \\ -I_o\left(\ddot{\varphi} + \ddot{\psi}\cos\vartheta - \dot{\psi}\dot{\vartheta}\sin\vartheta\right) \end{bmatrix}. \tag{2.16}$$

Use of the Cardanôs angles

As an alternative to the description of the spherical motion by means of Eulerôs angles we will present description using Cardanôs angles. Let us imagine that the movable system in Fig. 2.3 achieved the current position by means of three sequential rotations.

a) rotation about axis z -angle ψ

b) rotation about axis $x^{\backslash}$ -angle ϑ

c) rotation about axis $y^{\backslash\backslash}$ -angle φ

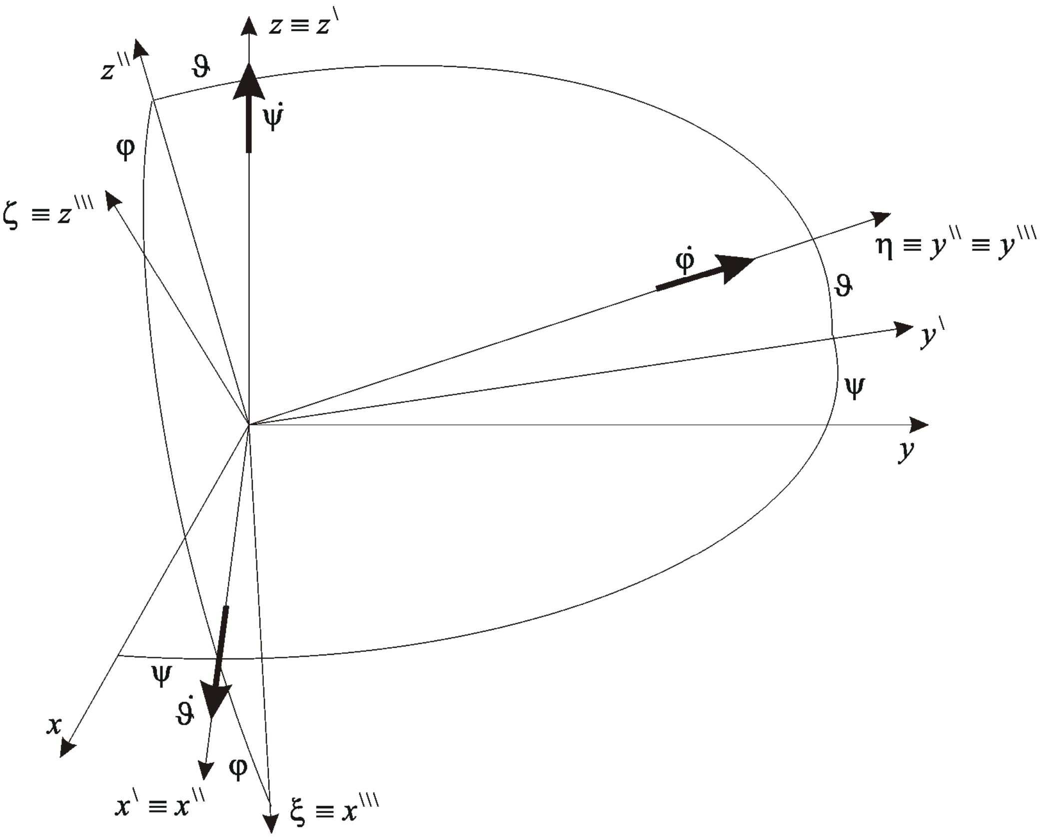

Fig. 2.3

The special case of modified Cardanôs angles for rotationally symmetric body will not be discussed. It can be said that those angles would be identical with modified Eulerôs angles but axis of own rotation would be identical with body axis of symmetry $\eta \equiv y^{\backslash\backslash}$. Following Fig. 2.3 we can write the vector of resulting angle speed expressed in the coordinate system ξ, η, ζ connected with moving body in form

$$\boldsymbol{\Upsilon}_{\xi\eta\zeta} = \begin{bmatrix} \dot{\vartheta}\cos\varphi - \dot{\psi}\cos\vartheta\sin\varphi \\ \dot{\varphi} + \dot{\psi}\sin\vartheta \\ \dot{\psi}\cos\vartheta\cos\varphi + \dot{\vartheta}\sin\varphi \end{bmatrix}. \tag{2.17}$$

Inertia matrix has form (1.1). Especially for rotationally symmetrical body with axis of symmetry $\eta \equiv y^{\backslash\backslash}$ this matrix has form

$$\mathbf{J}_{\xi\eta\zeta} = \mathbf{J}^{\backslash\backslash} = \begin{bmatrix} I, & 0, & 0 \\ 0, & I_o, & 0 \\ 0, & 0, & I \end{bmatrix}, \tag{2.18}$$

where the meaning of presented symbols was explained in connection with relations (2.6) and (2.7). Moment of momentum of symmetric body can be written as

$$\mathbf{L}^{\backslash\backslash} = \mathbf{J}_{\xi\eta\zeta}\boldsymbol{\varkappa}_{\xi\eta\zeta} = \mathbf{J}^{\backslash\backslash}\boldsymbol{\varkappa}_{\xi\eta\zeta} = \begin{bmatrix} I\left(\dot{\vartheta}\cos\varphi - \dot{\psi}\cos\vartheta\sin\varphi\right) \\ I_o(\dot{\varphi} + \dot{\psi}\sin\vartheta) \\ I\left(\dot{\psi}\cos\vartheta\cos\varphi + \dot{\vartheta}\sin\varphi\right) \end{bmatrix}. \tag{2.19}$$

Moment of inertia forces can be expressed according to relation (2.16), it means

$$\mathbf{M}_D = -\dot{\mathbf{L}}^{\backslash\backslash} - \mathbf{q}^{\backslash\backslash} \times \mathbf{L}^{\backslash\backslash}. \tag{2.20}$$

Having substituted we can come to

$$\mathbf{M}_D = \begin{bmatrix} I_o\left(\dot{\varphi}\dot{\psi}\cos\vartheta\cos\varphi + \dot{\psi}^2\sin\vartheta\cos\vartheta\cos\varphi + \dot{\varphi}\dot{\vartheta}\sin\varphi + \dot{\psi}\dot{\vartheta}\sin\vartheta\sin\varphi\right) - \\ - I\left(\ddot{\vartheta}\cos\varphi - \ddot{\psi}\cos\vartheta\sin\varphi + 2\dot{\psi}\dot{\vartheta}\sin\vartheta\sin\varphi + \dot{\psi}^2\sin\vartheta\cos\vartheta\cos\varphi\right) \\ \\ - I_o\left(\ddot{\varphi} + \ddot{\psi}\sin\vartheta + \dot{\psi}\dot{\vartheta}\cos\vartheta\right) \\ \\ - I_o\left(\dot{\varphi}\dot{\vartheta}\cos\varphi + \dot{\psi}\dot{\vartheta}\sin\vartheta\cos\varphi - \dot{\varphi}\dot{\psi}\cos\vartheta\sin\varphi - \dot{\psi}^2\sin\vartheta\cos\vartheta\sin\varphi\right) - \\ - I\left(\ddot{\psi}\cos\vartheta\cos\varphi - 2\dot{\psi}\dot{\vartheta}\sin\vartheta\cos\varphi + \dot{\psi}^2\sin\vartheta\cos\vartheta\sin\varphi + \ddot{\vartheta}\sin\varphi\right) \end{bmatrix}. \tag{2.21}$$

In this case of rotationally symmetric body with axis of symmetry $\eta \equiv y^{\backslash\backslash}$ we can substitute $\varphi = 0$ into (2.21). Then all inertia effects are expressed in Ătwo comma coordinate system ñdepicted in Fig. 2.3. It corresponds to the introduction of modified Cardanôs angles.

The examples on spherical motion

Example 2.1

Rolling of bevel pinion 3 on fixed taper disk (Fig. 2.4) is forced by means of driver 2 rotation about vertical axis o_{21} by constant angle speed ω_{21}. Determine kinetic energy and inertia effects acting on truncated taper of mass *m*. Top angle of the bevel pinion is β.

Inputs: $m,\ r_S,\ \beta,\ I_o,\ I,\ \omega_{21} = konst.$

Recommendation: Use the relation

$$\mathbf{M}_D = -\dot{\mathbf{L}}^{\backslash\backslash} - \mathbf{q}^{\backslash\backslash} \times \mathbf{L}^{\backslash\backslash} = \begin{bmatrix} - I\ddot{\vartheta} - I_o\dot{\varphi}\dot{\psi}\sin\vartheta - (I_o - I)\dot{\psi}^2\sin\vartheta\cos\vartheta \\ - I\ddot{\psi}\sin\vartheta - 2I\dot{\psi}\dot{\vartheta}\cos\vartheta + I_o\left(\dot{\varphi}\dot{\vartheta} + \dot{\psi}\dot{\vartheta}\cos\vartheta\right) \\ - I_o\left(\ddot{\varphi} + \ddot{\psi}\cos\vartheta - \dot{\psi}\dot{\vartheta}\sin\vartheta\right) \end{bmatrix}.$$

Solution:

Element of a ruled surface which is contact surface straight line of driving tapers is in fact instantaneous axis of resulting rotation, because each of its point has zeros momentary speed. Respecting the fact that angle speed $\dot{\psi} = \omega_{21}$ we can determine the secondary angle speed

$$\dot{\varphi} = \omega_{32} = \omega_{21}\, tg\beta. \tag{2.22}$$

Because $\vartheta = 90° = konst.$, $\dot{\varphi} = \omega_{32} = konst.$ a $\dot{\psi} = konst.$, the moment of inertia forces has form

$$\mathbf{M}_D = \begin{bmatrix} -I_o\dot{\varphi}\dot{\psi}\sin\vartheta \\ 0 \\ 0 \end{bmatrix}. \tag{2.23}$$

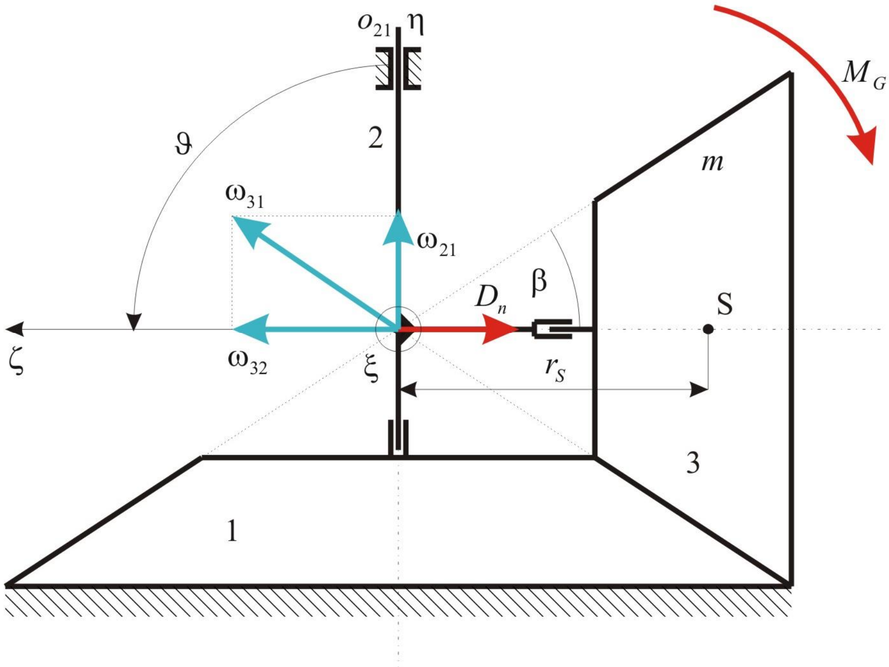

Fig. 2.4

This expression is called by gyroscopic moment which can be also expressed by means of vector relation (moment of inertia forces in this case does not contain any other terms)

$$\mathbf{M}_D = \mathbf{M}_G = I_o\dot{\boldsymbol{\varphi}}\times\dot{\boldsymbol{\psi}}. \tag{2.24}$$

The centrifugal force can be expressed in scalar or vector form, respectively (coordinates are related to Ătwo commaò system)

$$D_n = mr_S\omega_{21}^2, \qquad \text{resp.} \qquad \mathbf{D}_n = \begin{bmatrix} 0 \\ 0 \\ -mr_S\omega_{21}^2 \end{bmatrix}. \tag{2.25}$$

Example 2.2 [4]

Let us solve the forceless gyroscope motion Fig. 2.5. It means that gyroscope is not affected by any external forces and its gravity force is in balance with support reaction. This support is placed in a gyroscope centre of gravity. The initial angular speed ($t = 0$) of gyroscope is $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$.

Following the low of moment of momentum change in differential form ($\dot{\mathbf{L}} = \mathbf{M}$) we can say that $\dot{\mathbf{L}} = \mathbf{0} \Rightarrow \mathbf{L} = konst$. It means that moment of momentum vector has constant direction and magnitude. Both of these are unknown. Let us introduce the coordinate system ξ, η, ζ such as the ζ axis would be identical with axis of rotary symmetry of the gyroscope. The η axis lies in σ plane which is defined by ζ axis and vector $\boldsymbol{\omega}$ and simultaneously be perpendicular to the ζ axis. The ξ axis is perpendicular to the both axes and simultaneously the whole coordinate system is right-handed.

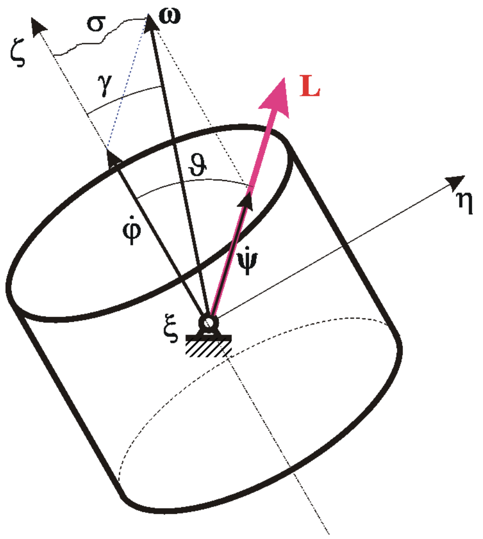

Fig. 2.5

As a start point of solution we can decompose angular speed vector $\boldsymbol{\omega}$ into two components in the ζ and $\mathbf{L}$ directions, respctively. Let us remind that this decomposition is imaginary only because we do not know a precession axis (axis of $\mathbf{L}$). Therefore it has to be proved that $\mathbf{L}$ vector lies in σ plane too. We express the inertia matrix and angular speed vector in ξ, η, ζ coordinate system having form

$$\mathbf{J}_{\xi\eta\zeta} = \mathbf{J}'' = \begin{bmatrix} I, & 0, & 0 \\ 0, & I, & 0 \\ 0, & 0, & I_o \end{bmatrix}, \quad \boldsymbol{\omega}_{\xi\eta\zeta} = \begin{bmatrix} 0 \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}. \tag{2.26}$$

Moment of momentum vector we can write down acording to the well known relation

$$\mathbf{L}^{\backslash\backslash} = \mathbf{J}^{\backslash\backslash}\boldsymbol{\Upsilon}_{\xi\eta\zeta} = \begin{bmatrix} 0 \\ I\omega_\eta \\ I_o\omega_\zeta \end{bmatrix}, \tag{2.27}$$

which shows that the moment of momentum component in the ξ direction is zero and for this reason this vector lies in $\eta\zeta$ plane. Now let us remind the (2.14) relation in form

$$\mathbf{L}^{\backslash\backslash} = \mathbf{J}^{\backslash\backslash}\boldsymbol{\Upsilon}_{\xi\eta\zeta} = \begin{bmatrix} I\dot{\vartheta} \\ I\dot{\psi}\sin\vartheta \\ I_o(\dot{\varphi}+\dot{\psi}\cos\vartheta) \end{bmatrix}.$$

Comparing both last relations we can write

$$L_\xi = I\dot{\vartheta} = 0 \Rightarrow \vartheta = \vartheta_0 = const. \tag{2.28}$$

From Fig. 2.5 and (2.14) follows

$$L_\eta = I\dot{\psi}\sin\vartheta_0 = L\sin\vartheta_0 \Rightarrow L = I\dot{\psi}. \tag{2.29}$$

Comparing (2.14) and (2.17) and respecting (2.28) we can come to relations

$$\omega_\eta = \dot{\psi}\sin\vartheta_0, \qquad \omega_\zeta = \dot{\varphi}+\dot{\psi}\cos\vartheta_0. \tag{2.30}$$

Because $\mathbf{L}^{\backslash\backslash} = \overrightarrow{const.} \Rightarrow L = const.$, where L is magnitude of the vector of moment of momentum $\mathbf{L}^{\backslash\backslash}$, and for this reason we can write

$$\dot{\psi} = \frac{L}{I} = \dot{\psi}_0. \tag{2.31}$$

Comparing ζ components of vector $\mathbf{L}^{\backslash\backslash}$ expressed by means of relations (2.27) and (2.14) we can come to

$$L_\zeta = L\cos\vartheta_0 = I_o(\dot{\varphi}+\dot{\psi}\cos\vartheta_0). \tag{2.32}$$

Respecting (2.31) the last equation can be solved for $\dot{\varphi}$ which takes a form

$$\dot{\varphi} = \dot{\varphi}_0 = \frac{L\cos\vartheta_0 - I_o\dot{\psi}_0\cos\vartheta_0}{I_o} = \frac{I\dot{\psi}_0\cos\vartheta_0 - I_o\dot{\psi}_0\cos\vartheta_0}{I_o} = \left(\frac{I}{I_0}-1\right)\dot{\psi}_0\cos\vartheta_0. \tag{2.33}$$

It follows from the last relation that the bracket will decide about the mutual sense of own rotation $\dot{\varphi}_0$ and precession $\dot{\psi}_0$. If the bracket sign will be negative $I_o > I$ than the own rotation sign will be different from the precession sign. In this case we can speak about the forward precession (assuming that $\vartheta_0 \in (0,90)°$). In the opposite case the precession is backward.

Sumarizing this knowledge we can come to $\vartheta = \vartheta_0, \dot{\psi} = \dot{\psi}_0, \dot{\varphi} = \dot{\varphi}_0$. Because γ is known angle (position of ζ and $\boldsymbol{\Upsilon}$ is known), we can from (2.30) and γ express angle ϑ_0. The rate of the components ω_ζ and ω_η is defined by known angle γ and this rate is function of the wanted angle ϑ_0. Substituting (2.33) for $\dot{\varphi}_0$ we can write

$$tg\gamma = \frac{\omega_\eta}{\omega_\zeta} = \frac{\dot\psi_0 \sin\vartheta_0}{\dot\varphi_0 + \dot\psi_0 \cos\vartheta_0} = \frac{I_o}{I} tg\,\vartheta_0 \Rightarrow tg\,\vartheta_0 = \frac{I}{I_o} tg\gamma. \tag{2.34}$$

As follows from the late relation the angle γ remains constant too. The angle ϑ_0 can be determined from the last relation. From the low of sines and from Fig. 2.6 we can compute $\dot\psi_0$ according to the relation

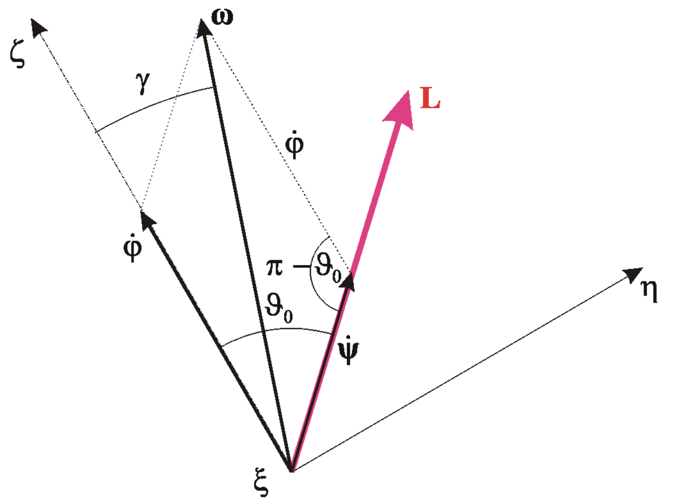

Fig. 2.6

$$\frac{\sin(\pi - \vartheta_0)}{\omega_0} = \frac{\sin\gamma}{\dot\psi_0} \;\Rightarrow\; \dot\psi_0 = \omega_0 \frac{\sin\gamma}{\sin\vartheta_0}. \tag{2.35}$$

Respecting

$$\sin\vartheta_0 = \frac{tg\,\vartheta_0}{\sqrt{1 + tg^2\vartheta_0}} \tag{2.36}$$

we can rewtite (2.35) into form

$$\dot\psi_0 = \omega_0 \frac{\sin\gamma}{\sin\vartheta_0} = \omega_0 \sqrt{\frac{I_o^2}{I^2}\cos^2\gamma + \sin^2\gamma}. \tag{2.37}$$

After substitution (2.34) into (2.33) for $\dot\varphi$ we have

$$\dot\varphi = \dot\varphi_0 = \omega_0 \frac{\sin\gamma}{\sin\vartheta_0}\left(\frac{I}{I_0} - 1\right)\cos\vartheta_0 = \omega_0 \frac{\sin\gamma}{tg\,\vartheta_0}\left(\frac{I}{I_0} - 1\right) = \omega_0 \frac{\sin\gamma}{\frac{I}{I_o} tg\gamma}\left(\frac{I}{I_0} - 1\right) = \omega_0 \cos\gamma\left(1 - \frac{I_o}{I}\right). \tag{2.38}$$

Taking into account the initial conditions $\varphi(0) = 0,\ \vartheta(0) = \vartheta_0,\ \psi(0) = 0,$ the time dependence of the individual angles can be expressed as

$$\psi(t) = \omega_0 t \sqrt{\frac{I_o^2}{I^2}\cos^2\gamma + \sin^2\gamma}, \qquad \varphi(t) = \omega_0 t \cos\gamma\left(1 - \frac{I_o}{I}\right). \tag{2.39}$$

It is clear from the last relations that the forceless gyroscope is able to move by constant own rotation and precession keeping constant nutation angle.

Example 2.3

Determine the inertia effects acting on cannon barrel having mass m and length l doing simultaneously primary precession corresponding to the azimuth angle ψ and nutation described by the elevation angle ϑ. The barrel is considered as a slender prismatic bar. Use the Cardanôs angles for computation.

The model of cannon barrel and its description by means of Cardanôs angles is depicted in Fig. 2.7. The angle speed of own rotation situated on the η axis is zero ($\dot{\varphi}=0$) and the corresponding rotating angle is considered also equal to zero. It means that $\varphi = 0$. Let us substitute $\varphi = 0,\ \dot{\varphi} = 0$ into (2.21) and than the moment of inertia forces can be written as

$$\mathbf{M}_D = \begin{bmatrix} (I_o - I)\dot{\psi}^2 \sin\vartheta\cos\vartheta - I\ddot{\vartheta} \\ -I_o\left(\ddot{\psi}\sin\vartheta + \dot{\psi}\dot{\vartheta}\cos\vartheta\right) \\ (2I - I_o)\dot{\psi}\dot{\vartheta}\sin\vartheta - I\ddot{\psi}\cos\vartheta \end{bmatrix}. \qquad (2.40)$$

To obtain a result inertia force it necessary to determine a centre of gravity acceleration. If the bar is prismatic and homogeneous the centre of gravity lies in half of length. Because the bar performs forward rotations about concurrent axes we can add up the angle speeds in the vector way. It is shown in Fig. 2.7

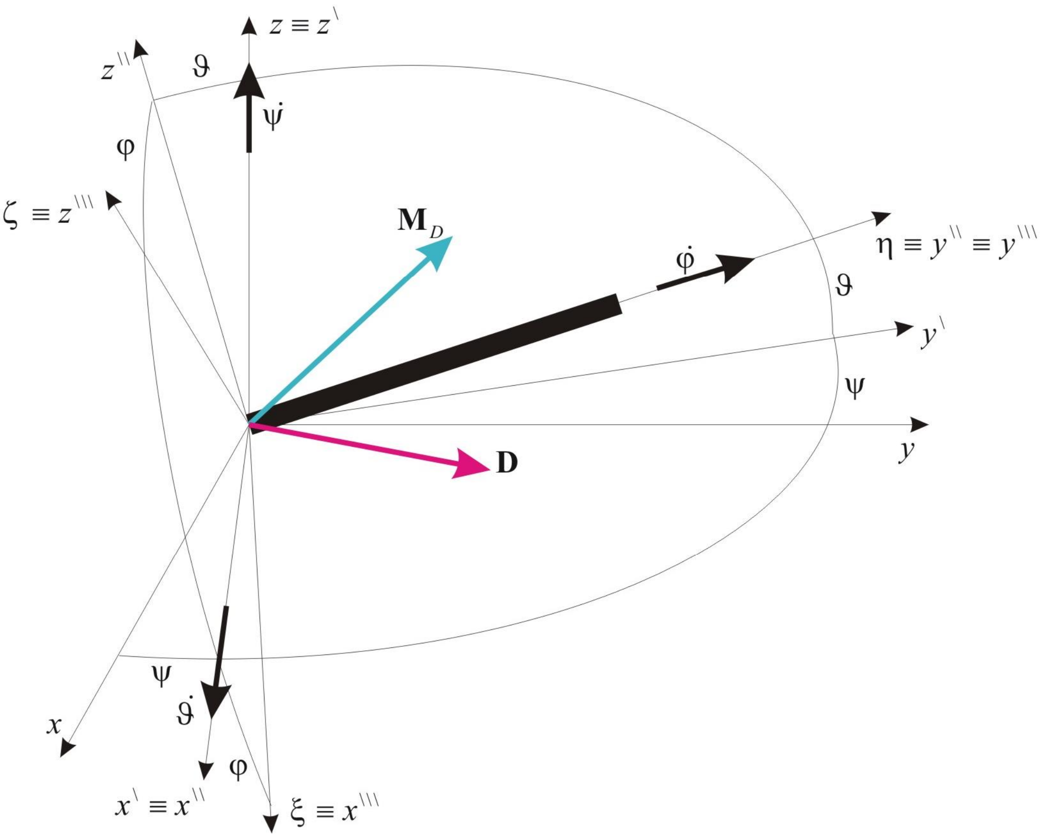

Fig. 2.7

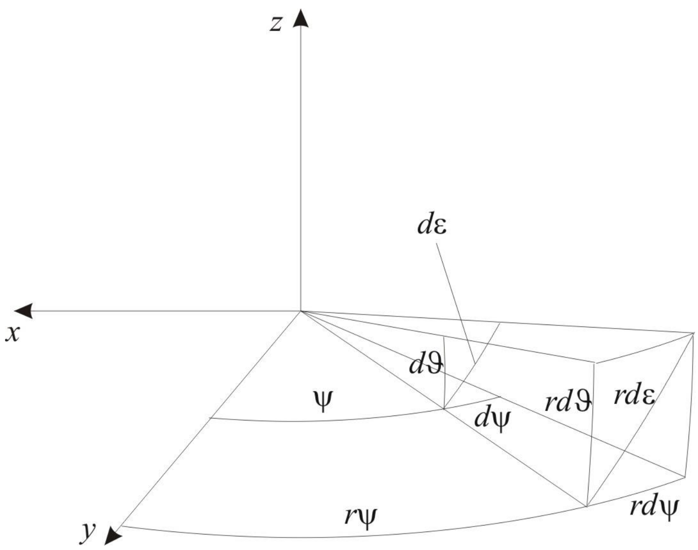

Fig. 2.8

According to the Pythagorasós low we can write

$$rd\varepsilon = \sqrt{r^2 d\psi^2 + r^2 d\vartheta^2}. \tag{2.41}$$

Having whole equation devided by rdt we can come to

$$\dot{\varepsilon} = \sqrt{\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\vartheta}{dt}\right)^2} = \sqrt{\dot{\psi}^2 + \dot{\vartheta}^2} \ . \tag{2.42}$$

It follows from the last relation that the resuling angle speed can be obtained as a vector product of individual angle speeds. Acceleration conditions will be a little bit more complicated. To keep generality of the example we will consider nonzero angle accelerations $\ddot{\psi}$ a $\ddot{\vartheta}$. For sake of simplicity let us mark $r = l/2$.

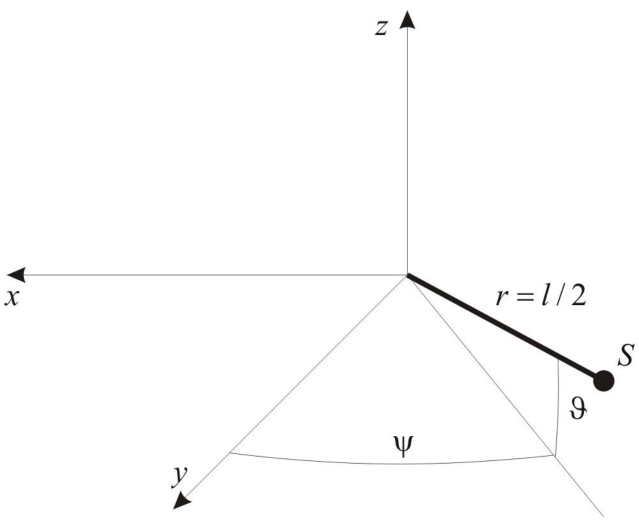

Fig. 2.9

Marking the radius-vector of center gravity by $\mathbf{r}_S$, we can write according Fig. 2.9

$$\mathbf{r}_S = \begin{bmatrix} -r\cos\vartheta\sin\psi \\ r\cos\vartheta\cos\psi \\ r\sin\vartheta \end{bmatrix},\ \mathbf{v}_S = \dot{\mathbf{r}}_S = \begin{bmatrix} r\dot{\vartheta}\sin\vartheta\sin\psi - r\dot{\psi}\cos\vartheta\cos\psi \\ -r\dot{\vartheta}\sin\vartheta\cos\psi - r\dot{\psi}\cos\vartheta\sin\psi \\ r\dot{\vartheta}\cos\vartheta \end{bmatrix}, \tag{2.43}$$

$$\mathbf{a}_S = \begin{bmatrix} r\ddot{\vartheta}\sin\vartheta\sin\psi + r\dot{\vartheta}^2\cos\vartheta\sin\psi + 2r\dot{\vartheta}\dot{\psi}\sin\vartheta\cos\psi - r\ddot{\psi}\cos\vartheta\cos\psi + r\dot{\psi}^2\cos\vartheta\sin\psi \\ -r\ddot{\vartheta}\sin\vartheta\cos\psi - r\dot{\vartheta}^2\cos\vartheta\cos\psi + 2r\dot{\vartheta}\dot{\psi}\sin\vartheta\sin\psi - r\ddot{\psi}\cos\vartheta\sin\psi - r\dot{\psi}^2\cos\vartheta\cos\psi \\ r\ddot{\vartheta}\cos\vartheta - r\dot{\vartheta}^2\sin\vartheta \end{bmatrix}. \tag{2.44}$$

This result can be obtained also in another way by the decomposition of resulting spherical motion to the primary precession 21 represented by angle ψ and secondary nutation 32 represented by angle ϑ. Than following relations can be written down

$$31 = 32 + 21,$$

$$\mathbf{v}_{31} = \mathbf{v}_{32} + \mathbf{v}_{21},\quad v_{32} = r\dot{\vartheta},\quad v_{21} = r\dot{\psi}\cos\vartheta,$$

$$\mathbf{a}_{31} = \mathbf{a}_{32} + \mathbf{a}_{21} + \mathbf{a}_c, \tag{2.45}$$

where

$$\mathbf{a}_{32} = \mathbf{a}_{n32} + \mathbf{a}_{t32},\quad \mathbf{a}_{21} = \mathbf{a}_{n21} + \mathbf{a}_{t21},\quad \mathbf{a}_c = 2\boldsymbol{\varkappa}_{21} \times \mathbf{v}_{32},\quad \boldsymbol{\varkappa}_{21} = \dot{\psi}\mathbf{k}. \tag{2.46}$$

The individual quantities have magnitudes

$$a_{n21} = r\dot{\psi}^2\cos\vartheta,\ a_{t21} = r\ddot{\psi}\cos\vartheta,\ a_{n32} = r\dot{\vartheta}^2,\ a_{r32} = r\ddot{\vartheta},\ a_c = 2r\dot{\psi}\dot{\vartheta}\sin\vartheta, \tag{2.47}$$

and directions depicted in Fig. 2.10

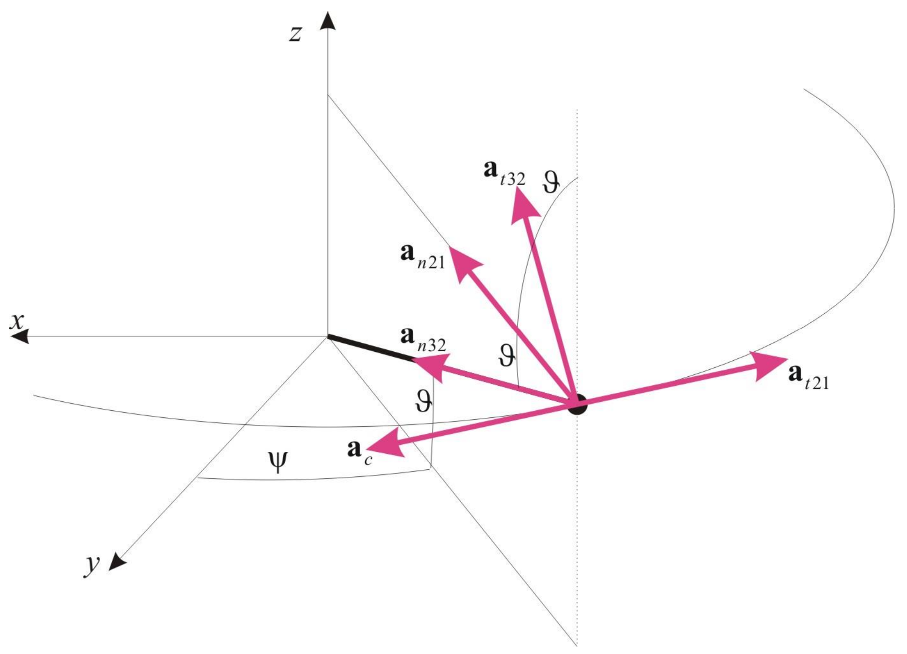

Fig. 2.10

These quantities can be written down in the vector way

$$\mathbf{a}_{n21} = \begin{bmatrix} r\dot{\psi}^2\cos\vartheta\sin\psi \\ -r\dot{\psi}^2\cos\vartheta\cos\psi \\ 0 \end{bmatrix},\ \mathbf{a}_{t21} = \begin{bmatrix} -r\ddot{\psi}\cos\vartheta\cos\psi \\ -r\ddot{\psi}\cos\vartheta\sin\psi \\ 0 \end{bmatrix},\ \mathbf{a}_{n32} = \begin{bmatrix} r\dot{\vartheta}^2\cos\vartheta\sin\psi \\ -r\dot{\vartheta}^2\cos\vartheta\cos\psi \\ -r\dot{\vartheta}^2\sin\vartheta \end{bmatrix}, \tag{2.48}$$

$$\mathbf{a}_{t32} = \begin{bmatrix} r\ddot{\vartheta}\sin\vartheta\sin\psi \\ -r\ddot{\vartheta}\sin\vartheta\cos\psi \\ r\ddot{\vartheta}\cos\vartheta \end{bmatrix},\ \mathbf{a}_c = \begin{bmatrix} 2r\dot{\vartheta}\dot{\psi}\sin\vartheta\cos\psi \\ 2r\dot{\vartheta}\dot{\psi}\sin\vartheta\sin\psi \\ 0 \end{bmatrix}. \tag{2.49}$$

The centre of gravity acceleration of the cannon barrel can be obtained as a vector summation of these five accelerations leading to (2.44). The resulting inertia force acting in the spherical motion centre will be equal to

$$\mathbf{D} = -m\mathbf{a}_S = -m(\mathbf{a}_{n21} + \mathbf{a}_{t21} + \mathbf{a}_{n32} + \mathbf{a}_{t32} + \mathbf{a}_c). \tag{2.50}$$

Note:

The direction of Coriolis acceleration is given by the vector product $\mathbf{a}_c = 2\boldsymbol{\omega}_{21} \times \mathbf{v}_{32}$, where $\omega_{21} = \dot{\psi}$ *and* $v_{32} = r\dot{\vartheta}$.

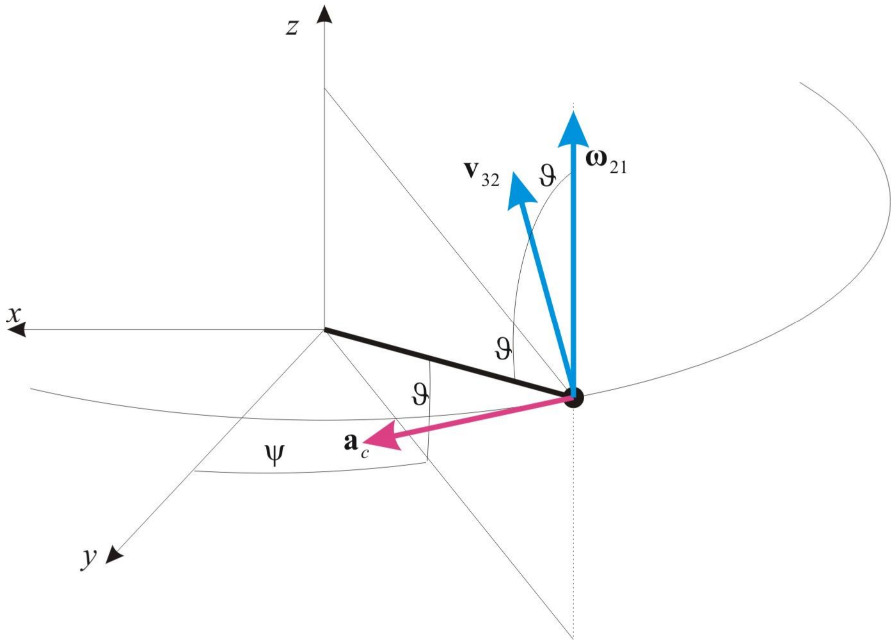

Fig. 2.11

It is convenient to remind that right hand fingers indicate direction of vector multiplication, it means from $\boldsymbol{\omega}_{21}$ *to* $\mathbf{v}_{32}$ *and thumb shows the direction of* ***Ă****result****ñ****(vector product).*

Example 2.4

As the next example let us present the rotating bar whose steady motion we solved in example 1.2. Now we will look for the solution of the unsteady motion which corresponds to spherical motion. The Fig. 2.12 shows the bar 3 jointed via the rotating couplings with massless member 2 exposed to acting of constant moment M_0. From Fig. 2.12 follows that $\varphi(0)=0, \dot{\varphi}(0)=0$, it means tha bar rotation about its own axis is zero. Let us assemble the equations of motion for the numerical solution of motion.

Input quantities: $I_0, I, M_0, \psi(0)=0, \dot{\psi}(0)=0, \ \ \vartheta(0)=\vartheta_0$ (small), $\dot{\vartheta}(0)=0.$

The coordinate system will be chosen in such a way that the bar symmetry axis will be identical with the ζ axis. Further ξ axis we place in such a way that nutation (by means of nutation the precession axis swichs to the axis of own rotation) will lie on the ξ axis. The third axis η should be completed to the right-handed coordinate system ξ, η, ζ. The equations of motion will be two moment conditions to the axes ξ and η. It can be written down in the vector form (Force conditions of dynamic equilibrium would serve to the computation of suspension reactions-they will be left out)

$$\mathbf{M}+\mathbf{M}_D=\mathbf{0}. \tag{2.51}$$

After expanding we get

$$\begin{aligned} &mg\frac{l}{2}\sin\vartheta + I\ddot{\vartheta} + (I_o - I)\,\dot{\psi}^2\sin\vartheta\cos\vartheta = 0,\\ &M_0\sin\vartheta + I\ddot{\psi}\sin\vartheta + (2I - I_o)\dot{\psi}\dot{\vartheta}\cos\vartheta = 0. \end{aligned} \tag{2.52}$$

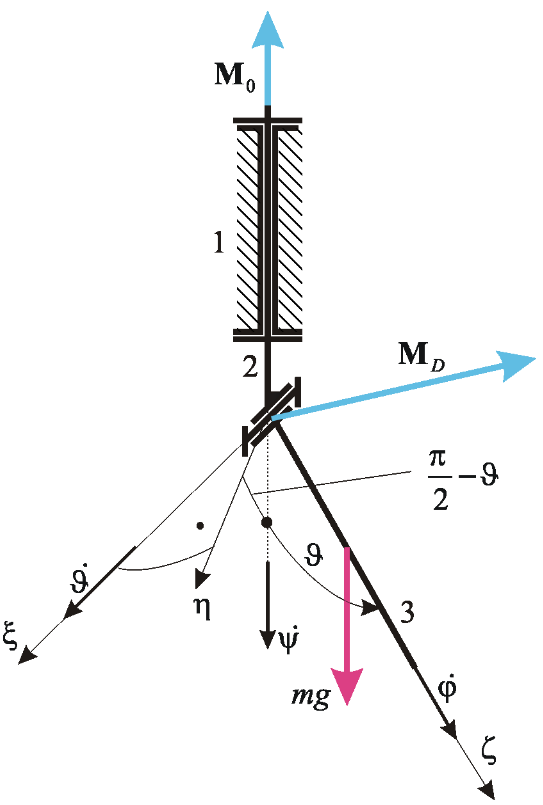

Fig. 2.12

The equations (2.52) can be rewritten into form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}(t) = \mathbf{p}(\mathbf{q},\dot{\mathbf{q}}), \tag{2.53}$$

where

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} I & 0 \\ 0 & I\sin\vartheta \end{bmatrix},\ \mathbf{q} = \begin{bmatrix} \vartheta \\ \psi \end{bmatrix},\ \mathbf{p}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} -mg\dfrac{l}{2}\sin\vartheta - (I_o - I)\dot{\psi}^2 \sin\vartheta\cos\vartheta \\ -M_0 \sin\vartheta - (2I - I_o)\dot{\psi}\dot{\vartheta} \end{bmatrix}. \tag{2.54}$$

This system of nonlinear differential equations can not be solved in the analytical way therefore only one way can be used for integration-numerical way. Employing standard procedure of MATLAB -*ode23* or *ode45* it is suitable to apply a certain arrangement. The presented procedures are convenient for solution of the matrix differential equation with initial condition

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x},t), \quad \mathbf{x}(0) = \mathbf{x}_0 . \tag{2.55}$$

To achieve the form (2.55) we add trivial identity

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}(t) - \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}(t) = \mathbf{0} \tag{2.56}$$

to the equation (2.53) and both equations can be written down in compact matrix form

$$\begin{bmatrix} \mathbf{0} & \mathbf{M}(\mathbf{q}) \\ \mathbf{M}(\mathbf{q}) & \mathbf{0} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}(\mathbf{q}) \end{bmatrix}\begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(\mathbf{q},\dot{\mathbf{q}}) \\ \mathbf{0} \end{bmatrix}, \tag{2.57}$$

which can be further rewritten as

$$\mathbf{A}(\mathbf{x})\,\dot{\mathbf{x}}(t) - \mathbf{B}(\mathbf{x})\,\mathbf{x}(t) = \mathbf{c}(\mathbf{x}). \tag{2.58}$$

The meaning of symbols $\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}), \mathbf{c}(\mathbf{x})$ follows from foregoing two relations. All these quantities are function of the state vector

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}. \tag{2.59}$$

The equation (2.58) can be converted into form (2.55) by pre-multiplying $\mathbf{A}^{-1}(\mathbf{q})$ and than we can come to

$$\dot{\mathbf{x}}(t) = \underbrace{\mathbf{A}^{-1}(\mathbf{x})[\mathbf{B}(\mathbf{x})\mathbf{x}(t) + \mathbf{c}(\mathbf{x})]}_{\mathbf{f}(\mathbf{x},t)}, \quad \mathbf{x}(0) = \mathbf{x}_0 . \tag{2.60}$$

Substituting concrete values

$$m = 1kg,\ l = 0.5\ m,\ I = \frac{ml^2}{3},\ I_o = I/20,\ \vartheta_0 = 0.01,\ \ M_0 = 10\ Nm$$

we can get results depicted in Figs. 2.13-2.16. The generalized displacements $\vartheta(t)$ and $\psi(t)$ are primarily represented and the generalized velocities $\dot{\vartheta}(t)$ and $\dot{\psi}(t)$ follow them.

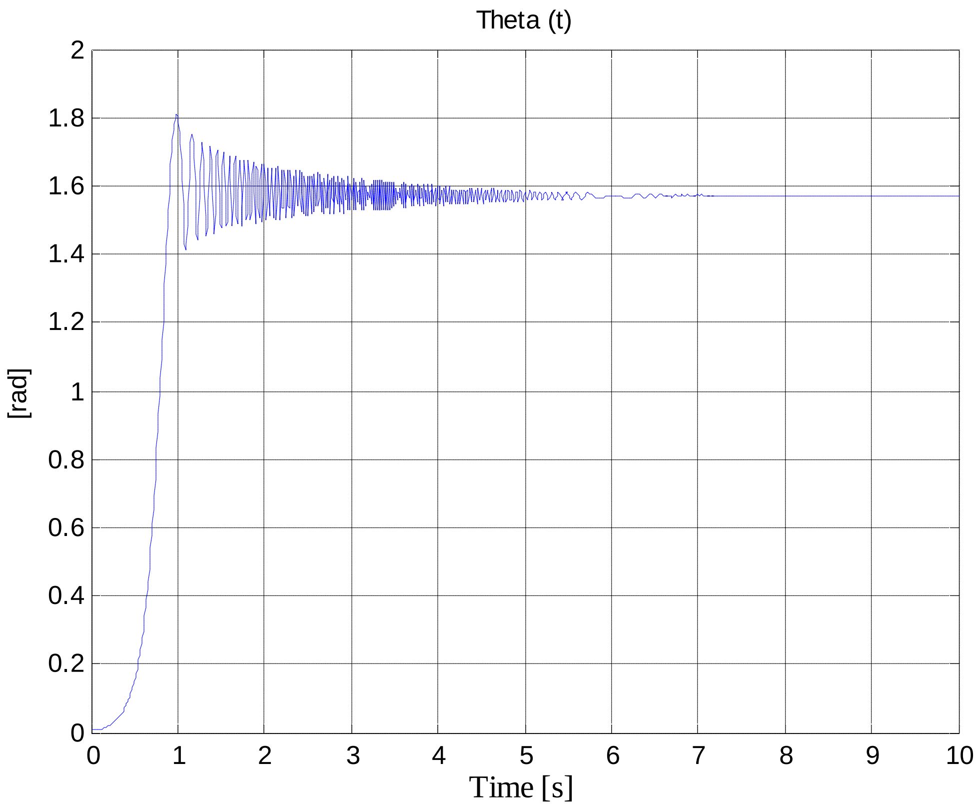

Fig. 2.13

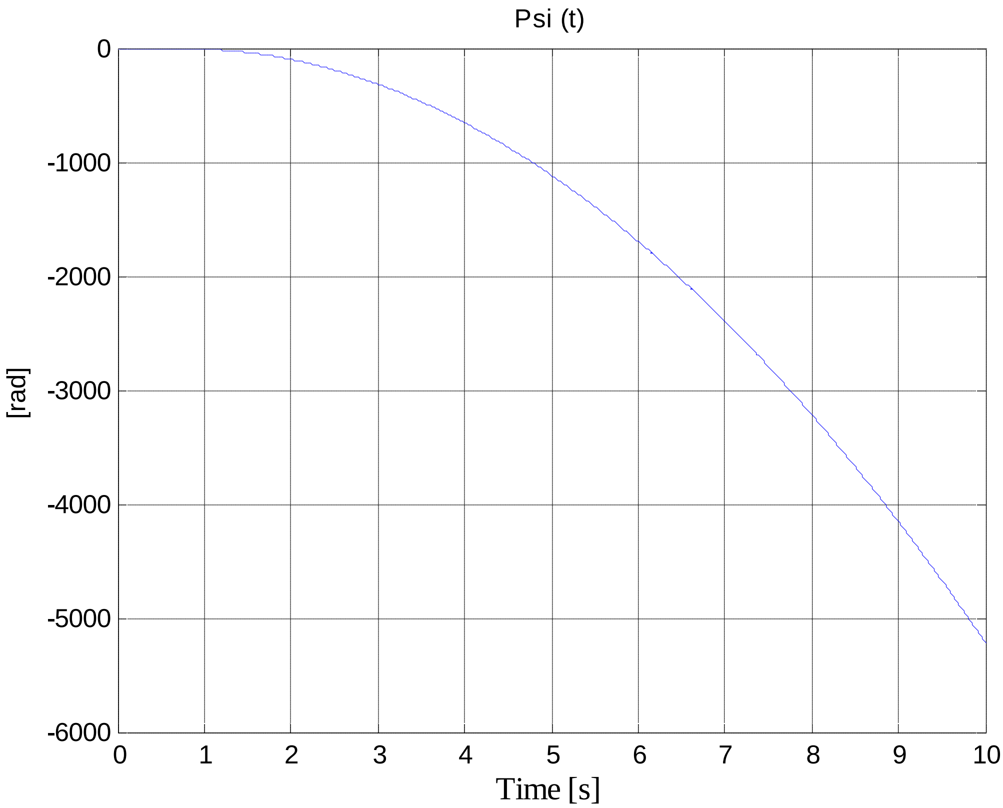

Fig. 2.14

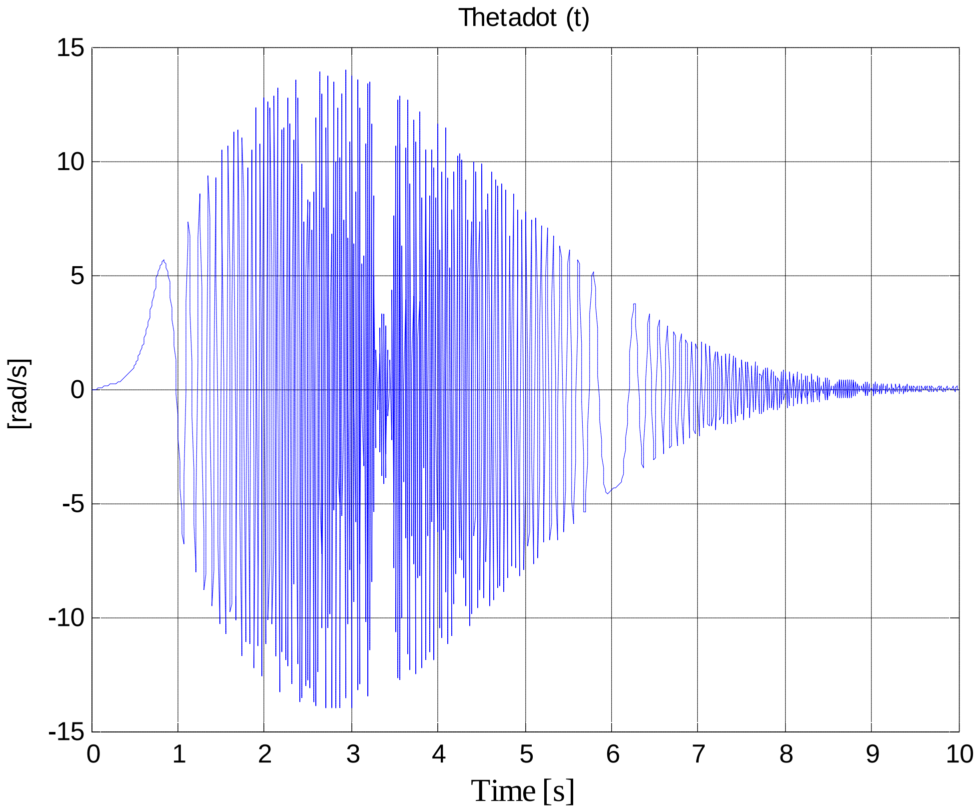

Fig. 2.15

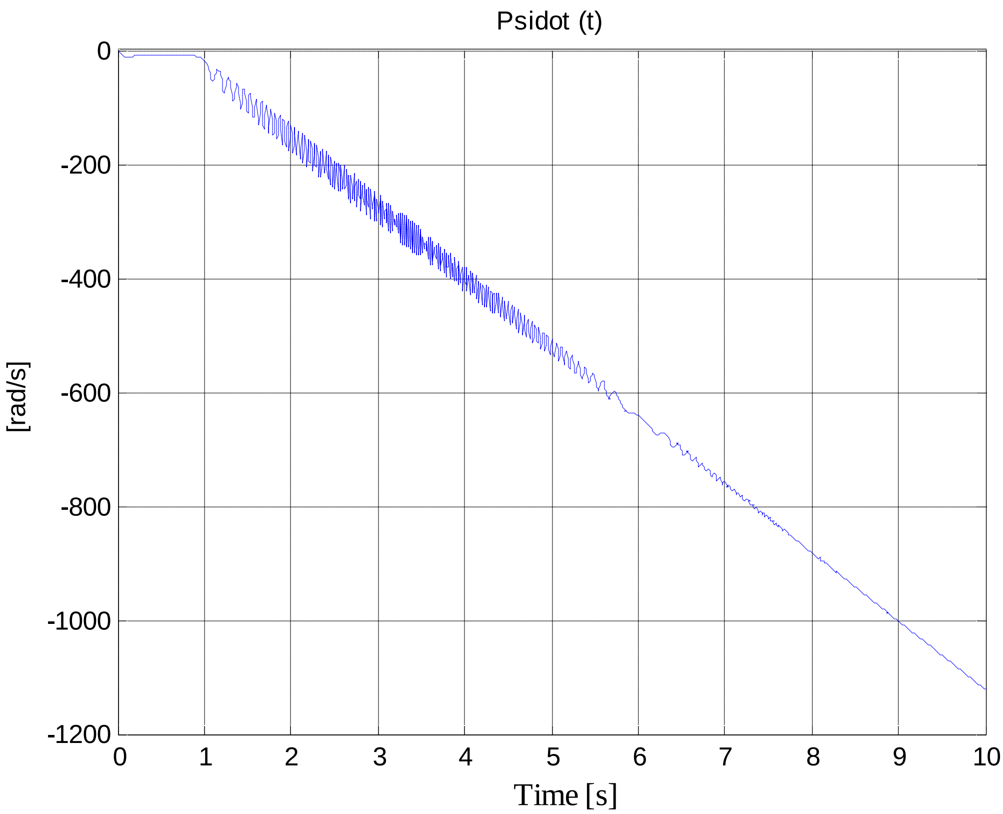

Fig. 2.16

Because the rotating bar is not braked and the environment resistance is neglected it is apparent that the limit value of nutation angle ϑ must be $\pi/2 = 1.5708$ which corresponds to the Fig. 2.13. Also the nutation speed $\dot{\vartheta}$ with steadying of angle ϑ is approaching to zero which corresponds to the Fig. 2.15

Example 2.5

Grinding pipe welds the abrasive wheel is doing spherical motion, see Fig. 2.17. Angular speed of specific rotation ω_0 corresponds to revolutions of the hand grinding machine $n = 7000\ RPM$. The precession angular speed is defined by the abrasive wheel center speed along the pipe perimeter $v_0 = 0.2\ m/s$. The nutation angle is permanently 90° so that nutation angular speed and acceleration are equal to zero. Let us determine the inertial effects acting on the abrasive wheel and on the hand of workman which operates with grinding machine. The pipe diameter is $D = 2R = 0.75\ m$.

Solution

Let us introduce the coordinate system according to the Fig. 2.17. The precession angle speed can be determinated as follows

$$\dot{\psi} = \frac{v_0}{R} = \frac{2v_0}{D} = 0.5333\ rad/s. \tag{2.61}$$

As the next step we obtain the specific rotation angle speed of the wheel

$$\dot{\varphi} = \omega_0 = \frac{\pi n}{30} = 733.0383\ rad/s. \tag{2.62}$$

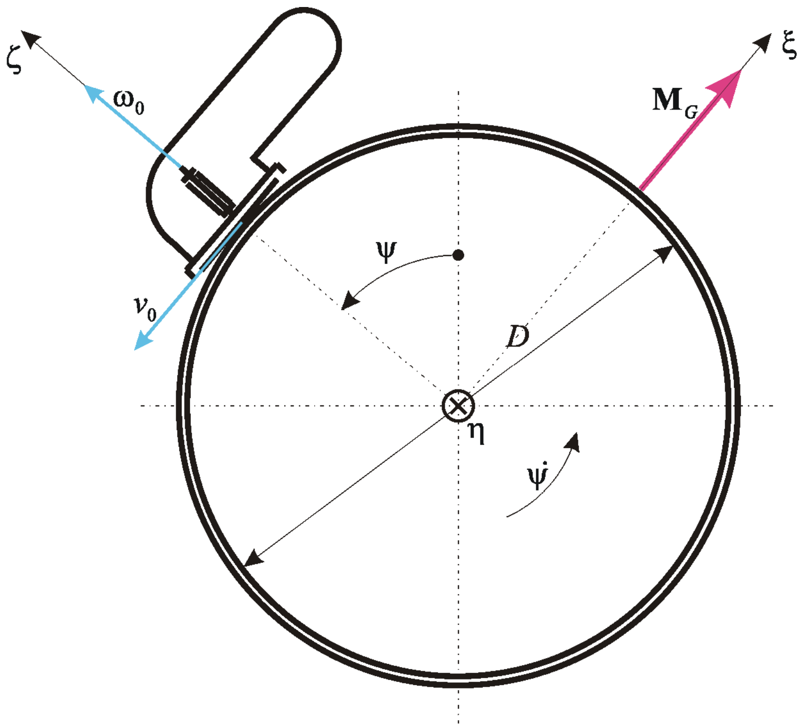

Fig. 2.17

We can neglect resultant inertia force acting on the wheel because the velocity v_0 is small and constant. For this reason the centrifugal force will be zero. From all components of moment of inertia forces only gyroscopic moment will be non-zero and takes a form

$$M_G = I_o \omega_0 \dot{\psi} = 0.0977\ Nm. \tag{2.63}$$

Example 2.6

Let us solve the motion of heavy gyroscope by means of

a) Euler’s modified angles

b) Cardan’s angles

Let us suppose that the specific rotation speed of the gyroscope $\dot{\varphi}$ is constant and much more larger then angle speeds of precession and nutation $\dot{\varphi} = \omega_0 >> \dot{\psi}, \dot{\vartheta}$. Further we will suppose that change of nutation angle is small, so that the nutation angle cen be expressed as sum of constant and small perturbation

$$\vartheta = \vartheta_0 + \Delta\vartheta,\ \ \Delta\vartheta << 1,\ \sin\vartheta \doteq \sin\vartheta_0 . \tag{2.64}$$

a) The Fig. 2.18 shows heavy gyroscope whose position is described by modified Euler’s angles

Corresponding initial condition are:

$$t = 0,\ \psi(0) = 0,\ \dot{\psi}(0) = 0,\ \varphi(0) = 0,\ \dot{\varphi}(0) = \omega_0,\ \vartheta(0) = \vartheta_0 = \frac{3\pi}{2},\ \dot{\vartheta}(0) = \Delta\dot{\vartheta}(0) = 0,\ \Delta\vartheta(0) = 0. \tag{2.65}$$

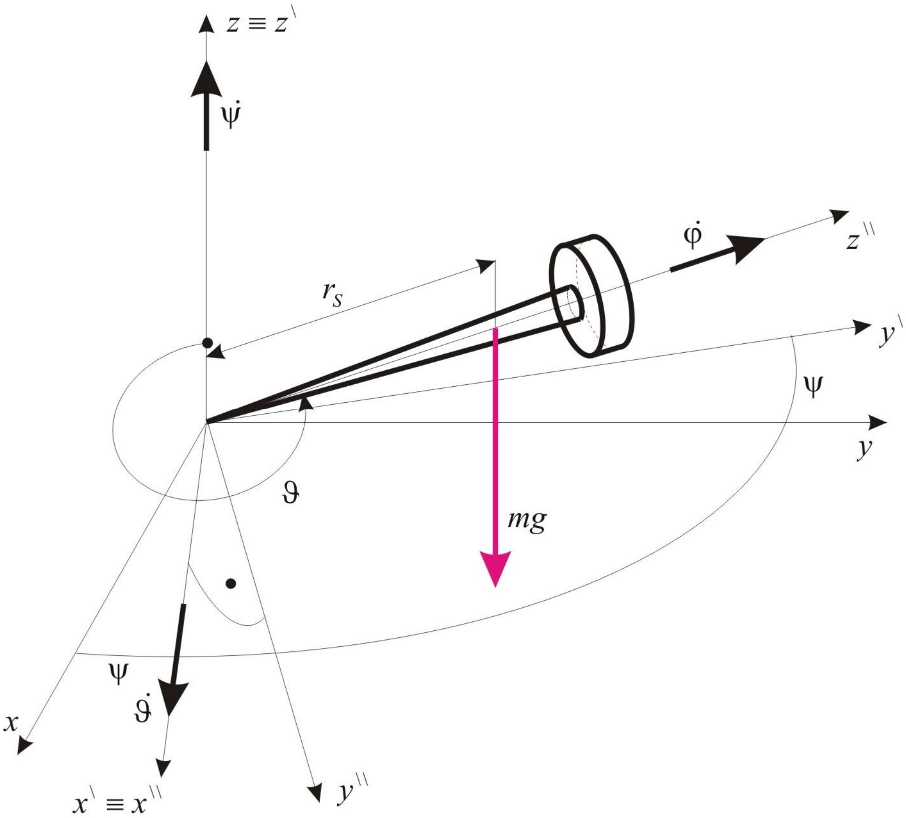

Fig. 2.18

Moment conditions of equilibrium to $x^{\backslash\backslash}$ and $y^{\backslash\backslash}$ axes briefly marked by ξ and η, respectively, can be written as

$$M_{D\xi} - mgr_S = 0,$$
$$M_{D\eta} = 0. \tag{2.66}$$

Moment equilibrium condition to the ζ axis was omited, because there holds a kinematic condition $\dot{\varphi} = \omega_0 = const$. Respecting (2.16) and $\sin\vartheta = \sin\vartheta_0 = -1$, $\dot{\varphi} = \omega_0$ we can write down these conditions (2.66) in form

$$I\Delta\ddot{\vartheta} - I_o\omega_0\dot{\psi} = -mgr_S,$$
$$I\ddot{\psi} + I_o\omega_0\Delta\dot{\vartheta} = 0. \tag{2.67}$$

b) The same gyroscope is depicted in Fig. 2.19 whose position is described by means Cardanôs angles.

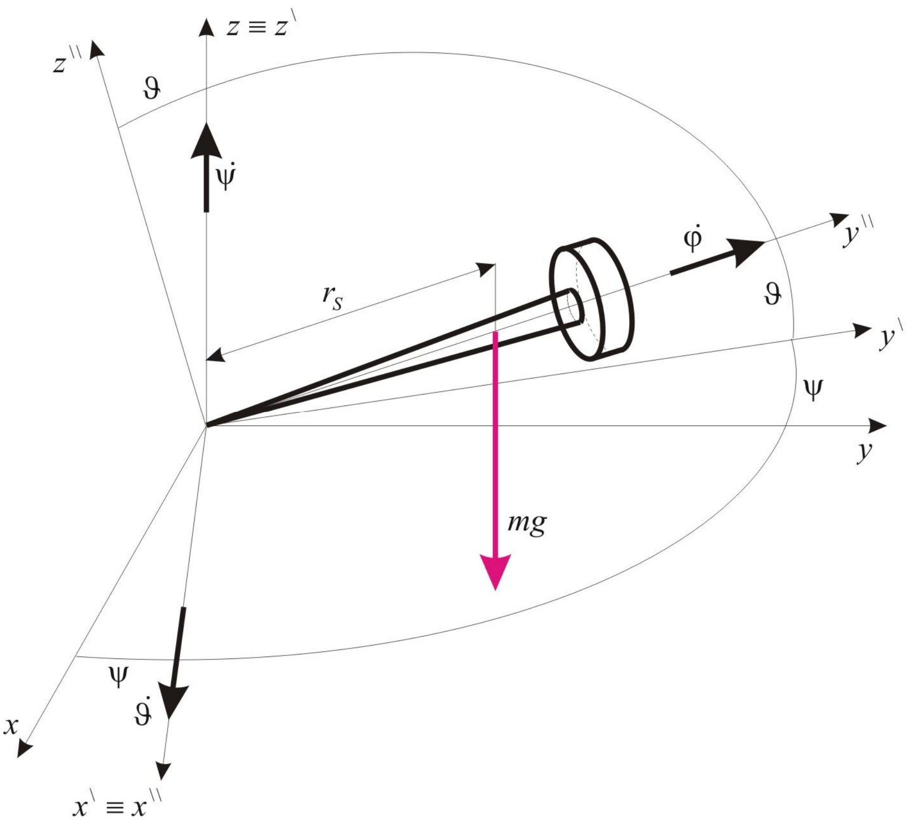

Fig. 2.19

Corresponding initial conditions are:

$$t = 0,\ \psi(0) = 0,\ \dot{\psi}(0) = 0,\ \varphi(0) = 0,\ \dot{\varphi}(0) = \omega_0,\ \vartheta(0) = \vartheta_0 = 0,\ \dot{\vartheta}(0) = \Delta\dot{\vartheta}(0) = 0,\ \Delta\vartheta(0) = 0. \tag{2.68}$$

For the sake of simplicity let us mark $\xi \equiv x^{\backslash\backslash}$, $\eta \equiv y^{\backslash\backslash}$, $\zeta \equiv z^{\backslash\backslash}$. Than the moment equilibrium condition to the axes ξ and ζ has form

$$\begin{aligned} &M_{D\xi} - mgr_S = 0, \\ &M_{D\zeta} = 0. \end{aligned} \tag{2.69}$$

Substituting moment of inertia forces (2.21) and respecting rotatory symmetry ($\varphi = 0$) we can come to

$$\begin{aligned} &I\,\Delta\ddot{\vartheta} - I_o\omega_0\dot{\psi} = -mgr_S, \\ &I\,\ddot{\psi} + I_o\omega_0\Delta\dot{\vartheta} = 0. \end{aligned} \tag{2.70}$$

Comparing both equations with (2.67) we can see that the results are identical. For solution we use the method of Laplace transformation. Let us mark firstly Laplace transform

$$T = \mathcal{L}\,\{\Delta\vartheta\},\;\; P = \mathcal{L}\,\{\psi\} \tag{2.71}$$

and apply it to the system (2.70). Then we can write

$$\begin{aligned} &Ip^2T + I_o\omega_0 pP = -mgr_S\frac{1}{p}, \\ &Ip^2P - I_o\omega_0 pT = 0. \end{aligned} \tag{2.72}$$

The expression $1/p$ on the right hand side of the first equation corresponds to Laplace transform of unit step. It means that the moment started acting in time $t = 0$. Solution of the system (2.72) takes a form

$$\begin{aligned} &T = \frac{-mgr_S}{Ip\left(p^2 + \Omega^2\right)}, \\ &P = \frac{-I_o\omega_0 mgr_S}{I^2}\frac{1}{p^2\left(p^2 + \Omega^2\right)}. \end{aligned} \tag{2.73}$$

In relation (2.73) we have used

$$\Omega^2 = \frac{I_o^2}{I^2}\omega_0^2. \tag{2.74}$$

Solution of the original system of equation (2.70) can be obtained by the inverse Laplace transform of (2.73) in form

$$\boxed{\begin{aligned} &\Delta\vartheta = -\frac{mgr_S}{I\Omega^2}(1 - \cos\Omega t), \\ &\psi = -\frac{mgr_S}{I_o\omega_0}\left(t - \frac{1}{\Omega}\sin\Omega t\right). \end{aligned}} \tag{2.75}$$

Example 2.7

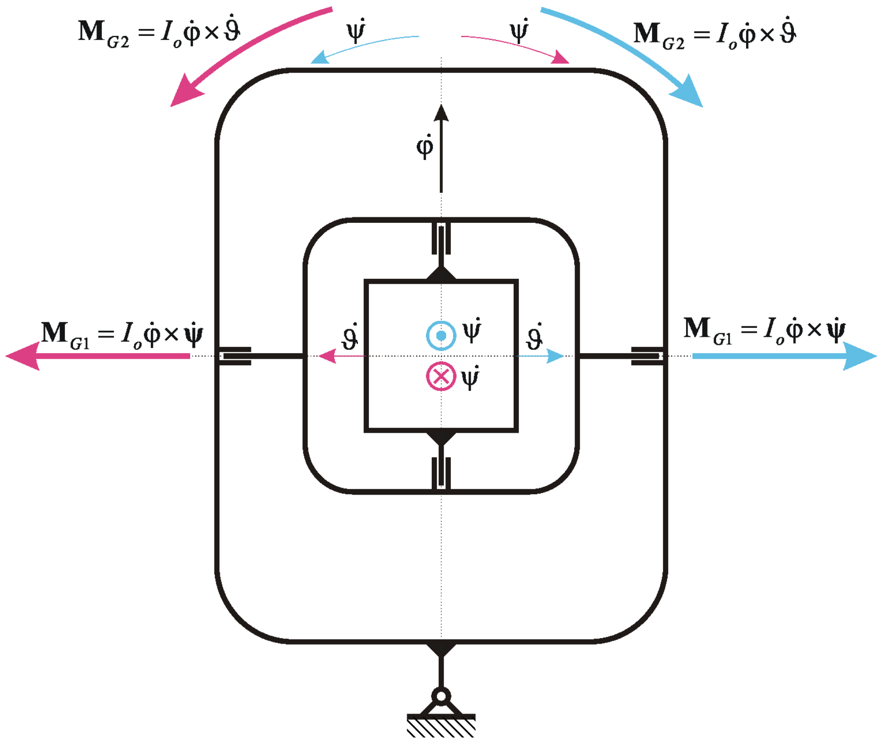

Fig. 2.20

The face look to the gyroscopic stabilizer is depicted in Fig. 2.20. It found an appllication in shipping traffic and was as curriosity used in railway traffic too. (The one rail train on the World exposition in Ósaca). Let us explain this stabilizing phenomenon.

Solution

The gyroscope angular speed is very high (tens of tausends RPM) represented by specific rotation speed $\dot{\varphi} = \omega_0$, which is considered as constant. Let us imagine the configuration in Fig. 2.20 which is marked by red color. The coach tilts to the right hand side which corresponds to the primary clockwise precession $\dot{\psi}$. The gyroscopic moment $\mathbf{M}_{G1}$ can be determinated from the vector product $\dot{\varphi}\times\dot{\mathbf{\gamma}}$ and it causes turning ϑ of the inner frame by angle speed $\dot{\vartheta}$ (all is marked by red color). The gyroscopic moment $\mathbf{M}_{G2}$ will be obtained from the vector product $\dot{\varphi}\times\dot{\vartheta}$ (M_{G1} and M_{G2} are labeled in (2.16)). This moment acts against the angle speed $\dot{\psi}$ direction and keeps coach from falling to the side. The oposite situation is marked by blue color.

3. General spatial and screw body motion

For investigation of spatial and screw body motion (screw motion is a special case of general spatial motion) we use a basic decomposition of the spatial motion into primary advance motion and secondary spherical motion. The result inertia effects can be then determinated as vector sum of inertia effects from these individual motions. (Because system is linear the principle of superposition can be used). Both of these individual motions and the corresponding inertia effects were discussed we can pass directly to the examples.

Examples on general spatial and screw body motion

Example 3.1

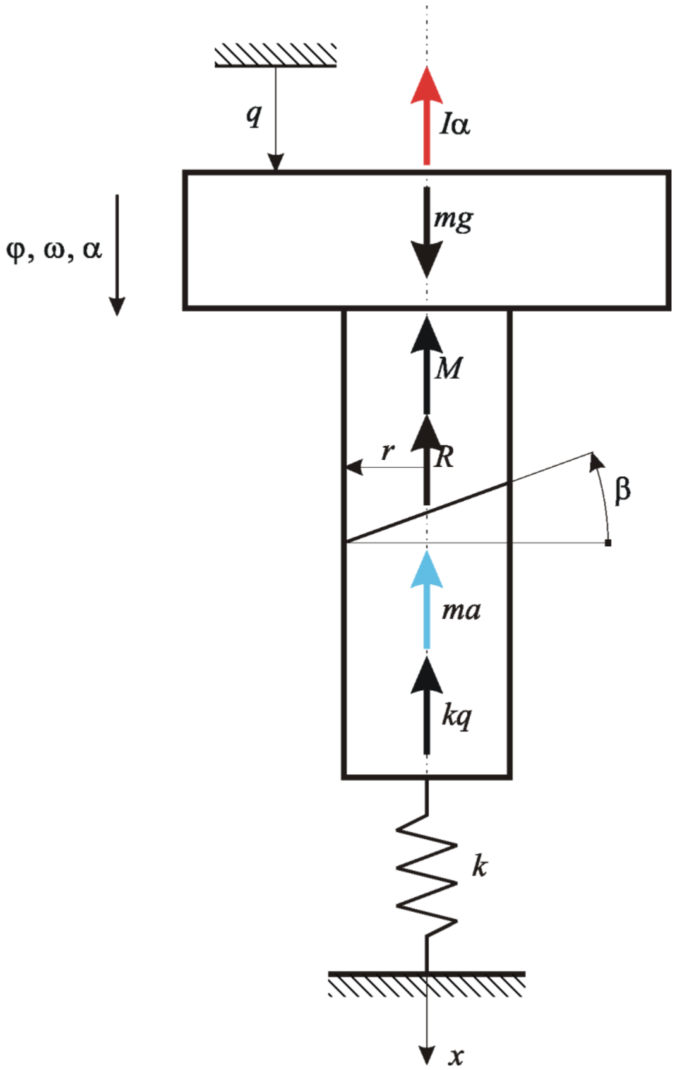

Fig. 3.1

The scheme of the screw (spindle) press doing screw motion from the start of piece pressing $(t = 0, q(0) = 0)$ is depicted in Fig. 3.1. For tha sake of simlicity we will suppose that piece material if perfectly elastic having total stiffness *k*. Our problem is to solve the press motion after achievment of contact with piece until maximal piece compression. Let us suppose that the spindle is not driven and it moves due only to own inertia. Displacement will be denoted from the place of first contact it means $q(0)=0$ and the initial velocity is defined by angular speed $\omega(0)=\dot{\varphi}(0)=\omega_0$. Respecting fact that the number of body degree motion is *1* its position is uniquely described by coordinate *q*. This motion consists of the primary motion in the axis *x* direction and secondary rotation about the same axis. The secondary rotation is coupled with the primary sliding motion due to the screw lead. The screw perimeter projected into cylinder surface is depicted in Fig. 3.2

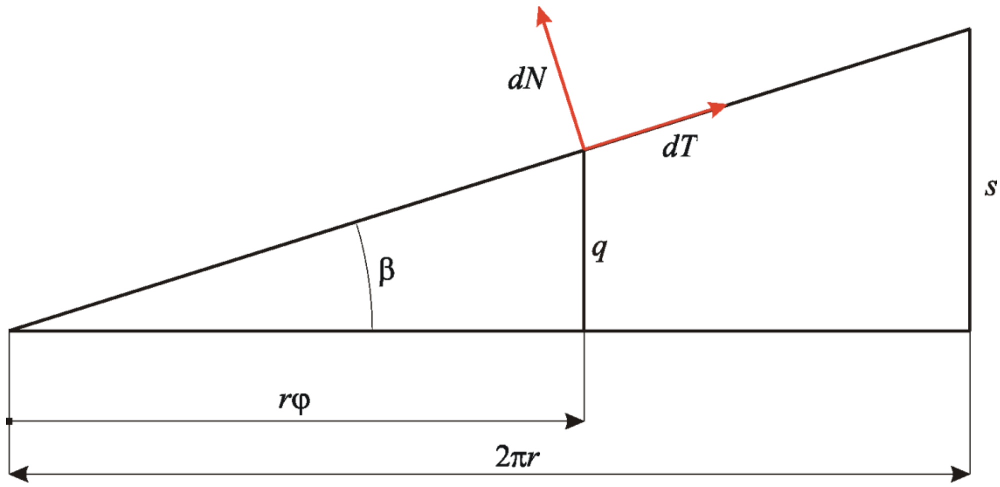

Fig. 3.2

From the triangle similarity we can determine the screw rotation dependence on the axial displacement *q* following relation

$$\varphi = 2\pi \frac{q}{s}, \tag{3.1}$$

where *s* is screw lead. Similar relations hold even for time derivatives.

$$\omega = \dot{\varphi} = 2\pi \frac{\dot{q}}{s},\ \alpha = \ddot{\varphi} = 2\pi \frac{\ddot{q}}{s}. \tag{3.2}$$

From the first relation (3.2) we can get the second initial (speed) condition

$$\dot{q}_0 = s\frac{\dot{\varphi}_0}{2\pi} = s\frac{\omega_0}{2\pi}. \tag{3.3}$$

Having assigned the lead angle it is possible to express this relation in form

$$\dot{q}_0 = r\dot{\varphi}_0 tg\beta.$$

In case of rotary symmetrical body the result inertia force from primary motion will take a form

$$D = ma = m\ddot{q}. \tag{3.4}$$

The result moment from elementary normal and friction forces in thread can be written in form

$$M = \int dT\, r\cos\beta - \int dN\, r\sin\beta = Nfr\cos\beta - Nr\sin\beta = Nr(\cos\beta - f\sin\beta), \tag{3.5}$$

the result force from these mentioned elementary forces in thread can be written in form

$$R = \int dN \cos\beta + \int dT \sin\beta = \int dN \cos\beta + \int dNf \sin\beta = N(\cos\beta + f \sin\beta). \tag{3.6}$$

Expressing N from (3.6)

$$N = \frac{R}{\cos\beta + f \sin\beta} \tag{3.7}$$

and substituting into (3.5) we can obtain dependence of acting force moment in thread on the result reaction in form

$$M = \frac{Rr(f\cos\beta - \sin\beta)}{\cos\beta + f \sin\beta} = \frac{Rr(\mathrm{tg}\tilde{\varphi}\cos\beta - \sin\beta)}{\cos\beta + \mathrm{tg}\tilde{\varphi}\sin\beta} = \frac{Rr(\mathrm{tg}\tilde{\varphi} - \mathrm{tg}\beta)}{1 + \mathrm{tg}\tilde{\varphi}\,\mathrm{tg}\beta}, \tag{3.8}$$

where $\tilde{\varphi}$ is friction angle satisfying condotion $f = \mathrm{tg}\tilde{\varphi}$. Using well known trigonometric formula

$$\mathrm{tg}(\alpha - \beta) = \frac{\mathrm{tg}\alpha - \mathrm{tg}\beta}{1 + \mathrm{tg}\alpha\,\mathrm{tg}\beta} \tag{3.9}$$

we can rearrange the relation (3.8) into form

$$M = -Rr\,\mathrm{tg}(\beta - \tilde{\varphi}). \tag{3.10}$$

The equations of motion describing screw press movement are represented by the force component condition in the axis direction and moment equilibrium condition about this axis. Both equations have form

$$R + m\ddot{q} + kq - mg = 0, \tag{3.11}$$

$$M + I\ddot{\varphi} = 0. \tag{3.12}$$

Let us substitute (3.10) into (3.12) and exclude R from equation (3.11). Then substituting R into (3.12) we can rewrite this equation and have

$$-(mg - m\ddot{q} - kq)r\,\mathrm{tg}(\beta - \tilde{\varphi}) + I\ddot{\varphi} = 0. \tag{3.13}$$

Angular acceleration $\ddot{\varphi}$ can be obtained by the second derivative of the relation (3.1). Then the equation (3.13) can be rewritten as (after canceling out r)

$$\left[m\mathrm{tg}(\beta - \tilde{\varphi}) + I\frac{2\pi}{sr}\right]\ddot{q}(t) + k\mathrm{tg}(\beta - \tilde{\varphi})q(t) = mg\,\mathrm{tg}(\beta - \tilde{\varphi}). \tag{3.14}$$

For the sake of notation simlicity let us mark

$$m_{red} = m\,\mathrm{tg}(\beta - \tilde{\varphi}) + I\frac{2\pi}{sr}, \qquad k_{red} = k\,\mathrm{tg}(\beta - \tilde{\varphi}), \qquad f_{red} = mg\,\mathrm{tg}(\beta - \tilde{\varphi}) \tag{3.15}$$

and let us rewrite the equation (3.14) into form

$$m_{red}\ddot{q}(t) + k_{red}q(t) = f_{red}. \tag{3.16}$$

Dividing by m_{red} the last equation takes form

$$\ddot{q}(t) + \Omega^2 q(t) = \frac{f_{red}}{m_{red}}. \tag{3.16a}$$

This equation describes vibration of the harmonic oscilator loaded by constant force f_{red} .

Solving maximal compression of the piece we can express acceleration from the equation (3.16)

$$a = \ddot{q} = \frac{1}{m_{red}}(f_{red} - k_{red} q) = a_0 - \Omega^2 q, \tag{3.17}$$

where $a_0 = \dfrac{f_{red}}{m_{red}} = \dfrac{mg\mathrm{tg}(\beta - \tilde{\varphi})}{m\mathrm{tg}(\beta - \tilde{\varphi}) + I\dfrac{2\pi}{sr}}$, $\Omega^2 = \dfrac{k_{red}}{m_{red}} = \dfrac{k\mathrm{tg}(\beta - \tilde{\varphi})}{m\mathrm{tg}(\beta - \tilde{\varphi}) + I\dfrac{2\pi}{sr}}$. (3.18)

This acceleration is dependent on the displacement. For the stop place detection of the press $q = q_{\max}$, $\dot{q} = 0$ we can use the basic kinematic expression for acceleration (v is press velocity in the axial direction)

$$a = \ddot{q} = \frac{d\left(v^2\right)}{2dq} = a_0 - \Omega^2 q. \tag{3.19}$$

Separation of variables can be performed in such a way that we multiply the equation (3.19) by dq and integrate

$$2\int_0^q \left(a_0 - \Omega^2 q\right) dq = \int_{v_0^2}^{v^2} d\left(v^2\right). \tag{3.20}$$

The relation between velocity and displacement follows from the last equation as

$$2a_0 q - \Omega^2 q^2 = v^2 - v_0^2. \tag{3.21}$$

For $v = \dot{q} = 0$ (3.21) represents the quadratic equation for $q_{\max}$ having form

$$\Omega^2 q_{\max}^2 - 2a_0 q_{\max} - v_0^2 = 0. \tag{3.22}$$

Its positive root is

$$q_{\max} = \frac{a_0 + \sqrt{a_0^2 + \Omega^2 v_0^2}}{\Omega^2}. \tag{3.23}$$

The initial velocity $v_0 = \dot{q}(0)$ was determinated from (3.3). The maximal compression determination is complete. The axial motion time dependence solution of the press means to find a solution of the differential equation of the second order (3.16a) which consists of the homogeneous solution (solution of (3.16) with zero right hand side)

$$q_H(t) = A\cos\Omega t + B\sin\Omega t \tag{3.24}$$

and particular solution which can be estimated according to the right hand side. It means in the form of constant

$$q_P = C. \tag{3.25}$$

This constant can be obtained by the substitution (3.25) into the equation with non-zero right hand side (3.16a)

$$\Omega^2 C = \frac{f_{red}}{m_{red}} \Rightarrow C = \frac{f_{red}}{m_{red}\Omega^2} = \frac{f_{red}}{k_{red}}. \tag{3.26}$$

The total solution then can be written as

$$q(t) = q_H(t) + q_P(t) = A\cos\Omega t + B\sin\Omega t + \frac{f_{red}}{k_{red}}. \tag{3.27}$$

Constants of integration A and B follow from initial conditions substituted into (3.27). For this reason let us derivate the equation (3.27) with respect to time and come to the relation for velocity

$$\dot{q}(t) = -\Omega A\sin\Omega t + \Omega B\cos\Omega t. \tag{3.28}$$

For time $t = 0$ the equations (3.27) and (3.28) take form

$$q(0) = q_0 = 0 = A + \frac{f_{red}}{k_{red}} \Rightarrow A = -\frac{f_{red}}{k_{red}}, \tag{3.29}$$

$$\dot{q}(0) = \dot{q}_0 = \Omega B \Rightarrow B = \frac{\dot{q}_0}{\Omega}. \tag{3.30}$$

Respecting (3.29) and (3.30) we can arrive at the solution (3.27) in form

$$q(t) = \frac{f_{red}}{k_{red}}(1 - \cos\Omega t) + \frac{\dot{q}_0}{\Omega}\sin\Omega t. \tag{3.31}$$

To express the solution in dependence on input parameters we take into account relations (3.3) and (3.15)

$$q(t) = \frac{mg}{k}(1 - \cos\Omega t) + s\frac{\omega_0}{2\pi\Omega}\sin\Omega t, \tag{3.32}$$

where

$$\Omega = \sqrt{\frac{k_{red}}{m_{red}}} = \sqrt{\frac{k\mathrm{tg}(\beta - \tilde{\varphi})}{m\mathrm{tg}(\beta - \tilde{\varphi}) + I\dfrac{2\pi}{sr}}} \tag{3.33}$$

is eigenfrequency of the system vibration. Taking into account non-linearity of friction forces in thread we have to respect solution (3.31) only in the span $q(t) \in \langle 0,\ q_{\max} \rangle$ where maximal displacement $q_{\max}$ is defined by (3.23).

Example 3.2

A jetplane flyes through the curve having radius R by velocity v_0. A plane turbine has a constant angular speed ω_0, mass m and axial moment of inertia I_o. Further the geometrical parameters of the turbine placement b and c are entered. The turbine mounting is depicted in Fig. 3.3. Our problem is to find out the reaction forces in turbine supports. Broadly speaking this problem is kinematic-static because all kinematic quantities are defined and goal of computation is determination of required forces.

To obtain a solution let us perform a basic decomposition of the motion with reference point in centre of gravity of the turbine. Let us imagine that the turbine moves by sliding motion having constant velocity of gravity centre v_0 along circle of radius R it means without any rotation. For this reason the tangential acceleration is zero (mass point motion along then circle of radius R). Remaining acceleration corresponds to normal acceleration of the gravity centre taking value

$$a_n = \frac{v_0^2}{R} = R\dot{\psi}^2, \qquad \left(\dot{\psi} = \frac{v_0}{R}\right). \tag{3.34}$$

Corresponding inertia force can be expressed as

$$D = mR\dot{\psi}^2 = m\frac{v_0^2}{R}, \tag{3.35}$$

which is marked by blue colour in Fig. 3.3. Now we can compose the primary sliding motion and the secondary spherical motion. Respecting fact that the secondary spherical motion centre is identical with gravity centre of turbine the tangential and normal inertia force corresponding to spherical motion will be equal to zero. Spherical motion is defined by primary precession velocity

$$\dot{\psi} = \frac{v_0}{R}, \tag{3.36}$$

whose vector has direction from the picture plane (right hand rule). The secondary rotation corresponds to the angular speed of turbine

$$\dot{\varphi} = \omega_0 = konst., \tag{3.37}$$

whose direction follows from the picture. Nutation angle ϑ lies between the specific rotation vector and precession vector it means 90°. As the final step we can determine a moment of inertia forces. This moment was developed in Chap. 2 by relation (2.16). All components of this moment are zero except for gyroscopic moment taking form

$$\overrightarrow{M} = I_o \vec{\dot{\varphi}} \times \vec{\dot{\psi}} \quad \Rightarrow \quad M_G = I_o \omega_0 \dot{\psi} = I_o \omega_0 \frac{v_0}{R}. \tag{3.38}$$

Gyroscopic moment direction is depicted by red arrow in Fig. 3.3

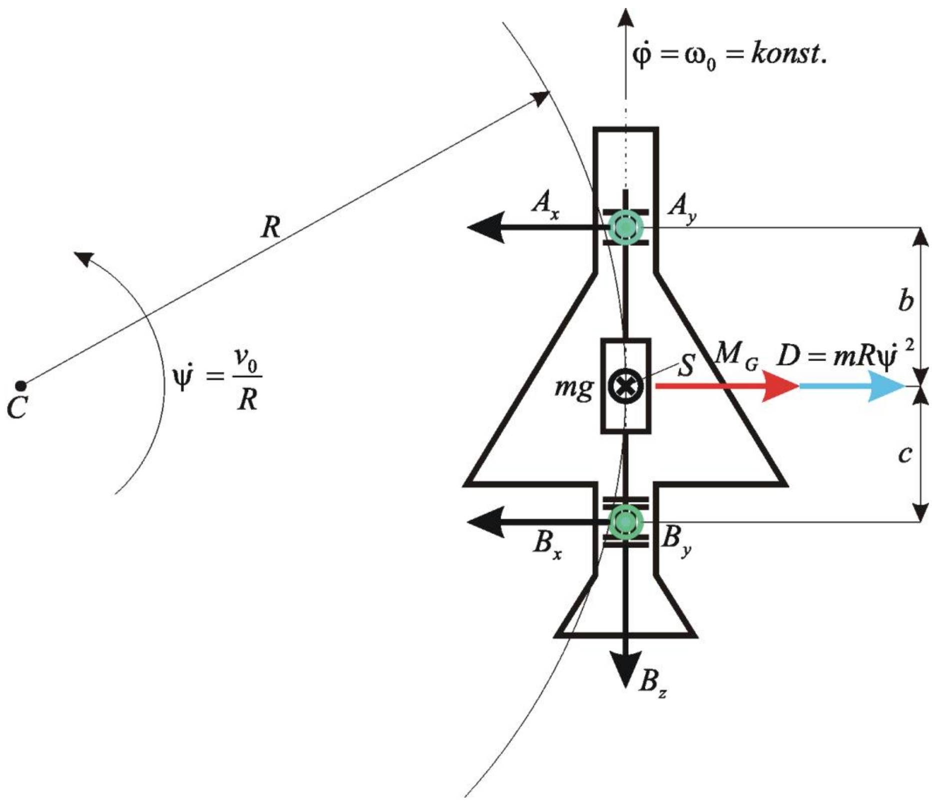

Fig. 3.3

The vectors directed to and from picture plane were marked by cross and dot in circle, respectively. This marking is being used commonly in electrical engineering. Because the body has 6 degrees of freedom (DOF) in space we can write 6 equilibrium conditions of dynamic balance. Really, the number of conditions will be 5. One condition is prescribed kinematic quantity $\dot{\varphi} = \omega_0 = konst$. Remaining 5 conditions have form (moment equilibrium conditions are related to the axes going through the gravity centre of turbine).

$$
\begin{aligned}
&B_z = 0, \\
&A_x + B_x - D = 0, \\
&A_y + B_y - mg = 0, \\
&A_x b - B_x c = 0, \\
&A_y b - B_y c + M_G = 0.
\end{aligned}
\tag{3.39}
$$

Substituting for D and M_G from (3.35) and (3.38) into (3.39) we can come to 5 algebraic equations for 5 unknown values of coupling forces in turbine bearings A_x, A_y, B_x, B_y, B_z. The task solution is complete.

Example 3.3

The next problem is again so called kinematic-static task where all of kinematic quantities are defined. The goal of our efforts is determination of inertia effects. The gyroscope doing two simultaneous rotations about two skew axes z and ζ is depicted in Fig. 3.4. The corresponding angular speeds Ω_0 and ω_0 are constant. The angle δ between ζ axis and its top view projection (Fig. 3.4) is constant too.

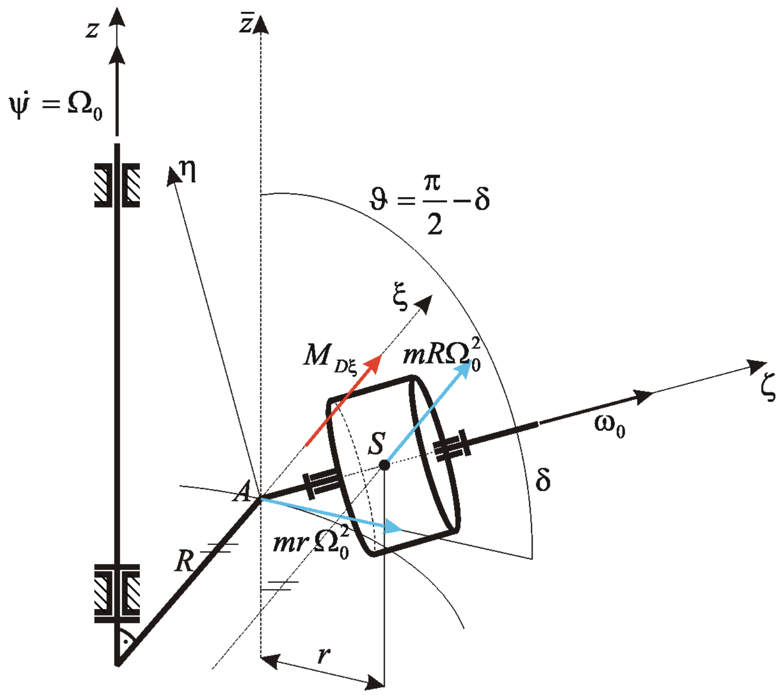

Fig. 3.4

The result is general spatial motion. The inertia effect investigation can be performed by means of basic decomposition, it means the decomposition into the primary sliding motion by velocity and acceleration of the reference point A and secondary spherical rotation about this point. This spherical rotation can be expressed by primary precession Ω_0 transferred into the $\bar{z}$ axis and secondary specific rotation ω_0 about the ζ axis. The ζ axis is identical with the specific rotation axis. The ξ is defined in such a way that the $\bar{z}$ axis turning slightly about the ξ axis comes to the ζ axis and the last η axis can be completed to the right-handed coordinate system ξ, η, ζ. As follows from the Fig. 3.4 the nutation angle ϑ is constant complement of the δ angle to 90°.

a) the primary motion is sliding

Let us imagine that the gyroscope is doing primary sliding motion by velocity and acceleration of reference point A. Because Ω_0 is constant the A point has only normal non-zero acceleration

$$a_{nA} = R\Omega_0^2, \qquad a_{tA} = 0. \tag{3.40}$$

Corresponding inertia force takes a form

$$D_{nU} = mR\Omega_0^2, \tag{3.41}$$

whose orientation and operation point are depicted in Fig. 3.4. Let us remind that the operation point of result inertia force of the body doing sliding motion is identical with body centre of gravity. Respecting (3.40) the tangetial force corresponding to the primary sliding motion is zero.

b) the secondary spherical motion superimposed to the primary sliding motion. It means that primary precession $\dot{\psi} = \Omega_0$ is transferred from the z to the $\bar{z}$ axis. The inertia force corresponding to the primary precession is acting in the spherical motion centre A and has a direction of normal inertia force from the rotation motion about $\bar{z}$ axis having angular speed Ω_0. It means perpendicular to $\bar{z}$ axis going through the gyroscope centre of gravity. Its magnitude is proportional to the perpendicular distance of the gyroscope gravity centre from $\bar{z}$ axis r and it can be expressed in form

$$D_{nS} = mr\Omega_0^2. \tag{3.42}$$

Tangential inertia force caused by primary precession is in result of constant angular speed Ω_0 equal to zero. The moment of inertia force corresponding to secondary spherical rotation remains to be determined. Following (3.38) only one componernt is non-zero while the other components are zero. It means that

$$M_{D\xi} = -I_o\omega_0 \underbrace{\sin\vartheta}_{\cos\delta} - (I_o - I_1)\Omega_0^2 \underbrace{\sin\vartheta\cos\vartheta}_{\cos\delta\sin\delta}, \quad M_{D\eta} = M_{D\zeta} = 0, \tag{3.43}$$

where I_1 is gyroscope moment of inertia with respect to axes ξ and η. The first term in the first relation (3.43) represents again gyroscopic moment which can be expressed as

$$\mathbf{M}_G = I_o\dot{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\psi}} = I_o\boldsymbol{\omega}_0 \times \boldsymbol{\Omega}_0 \quad \Rightarrow \quad M_G = I_o\omega_0\Omega_0 \sin\vartheta = I_o\omega_0\Omega_0\cos\delta. \tag{3.44}$$

It means that all inertia effects depicted in Fig. 3.4 are explained.

4. Basics of analytical mechanics
4.1. Classification of mechanical systems

The term Ągeneralized coordinatesñ will in this textbook present the coordinates defining positions of body or body system regardless these coordinates are longitudinal or angular and regardless connection of their number with DOF number *n*. The number of DOF expresses number of body or body system independent motions. The system description can be divided into two groups:

a) The system position is described by $m > n$ of generalized coordinates. Then $r = m - n$ coupling conditions (briefly couplings) have to be provided. As an example we can present body depicted in Fig. 4.1

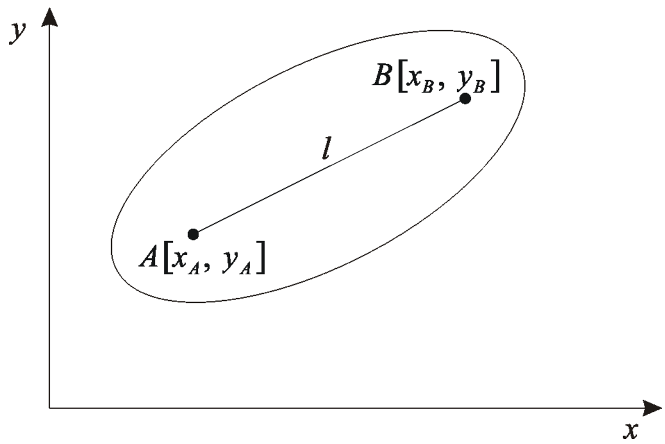

Fig. 4.1

Its position is described by four generalized coordinates $x_A,\ y_A, x_B,\ y_B$. It is well known that the body has $n = 3$ DOF. For this reason these coordianates have to be coupled by the condition of constant distance

$$(x_B - x_A)^2 + (y_B - y_A)^2 = l^2. \tag{4.1}$$

It means that $m = 4,\ n = 3$ and $r = m - n = 1.$ In opposite case without coupling condition the distance *A* and *B* would not be constant.

b) The system position is described by *n* generalized coordinates number of which correspond to number of DOF (driving coordinates).

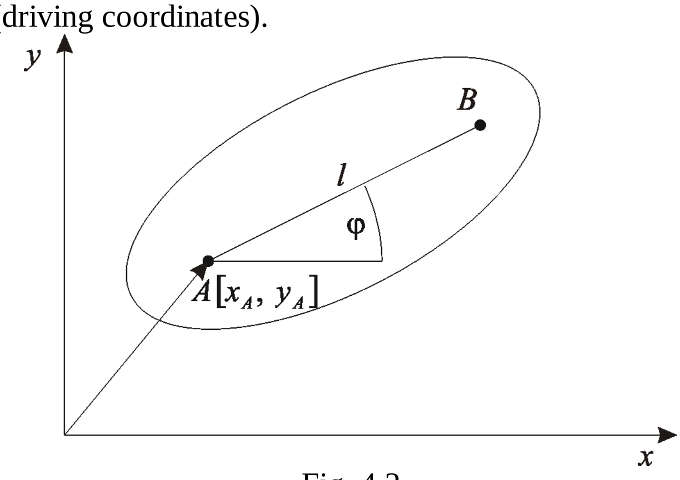

Fig. 4.2

As a suitable example can be presented the body depicted in Fig. 4.2 whose position is unambiguously described by three driving generalized coordinates x_A, y_A and φ. In this case $m = n = 3$ and number of coupling condition is $r = m - n = 0$.

According to the number of DOF we can divide systems into two groups
a) continuous (continuum) having $n \to \infty$ DOF number and
b) discrete having $n < \infty$ DOF number

For the next classification let us present three kinds of systems whose position will be described by redundant number of DOF $m > n$ and by corresponding coupling conditions.

1) The mechanism with harmonic cam. The attribute Ăharmonicñ follows from the coupling condition.

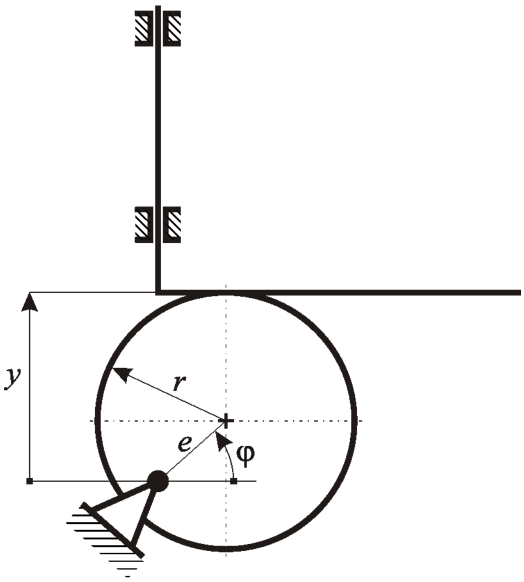

Fig. 4.3

Its position is described by generalized coordinates φ and y but the number of DOF is equal to 1 as can be proved. Therefore there is a coupling equation between those coordinates. This equation can be expressed both in integral and differential form, respectively. Integral and differential forms have form

$$y = e \sin \varphi + r, \quad \text{resp.} \quad d\,y = e \cos\varphi\, d\varphi. \tag{4.2}$$

Let us notice that coupling conditions do not contain explicitly time in nor integral neither differential form.

2) As the next example let us show the system depicted in Fig. 4.4. It is a pendulum with varying length of suspension. The system has $n = 2$ DOF which can be expressed by coordinates $r(t)$ a $\varphi(t)$. When the motion $r(t)$ is prescribed the excitation will be kinematic. When the system position is described by $m = 3$ coordinates $r(t), x, y$, see Fig. 4.4, it is necessary to add still one condition ($r = m - n = 3 - 2 = 1$), which means that the fiber is perfectly rigid and has constant length l_0

$$x^2 + y^2 - [l_0 - r(t) - b]^2 = 0. \tag{4.3}$$

The coupling condition(4.3) is in integral form. Its differential form can be written as

$$x\,dx + y\,dy - [l_0 - r(t) - b]\underbrace{\dot{r}(t)dt}_{d\,r(t)} = 0. \tag{4.4}$$

When the coordinate $r(t)$ has prescribed shape eg. in form $r(t) = r_0 \sin \omega t$, then equation (4.4) takes form

$$x\,dx + y\,dy + [l_0 - r_0 \sin \omega t - b] r_0 \omega \cos \omega t\, dt = 0. \tag{4.5}$$

Let us notice that both forms (4.4) and (4.5) contain explicitly time.

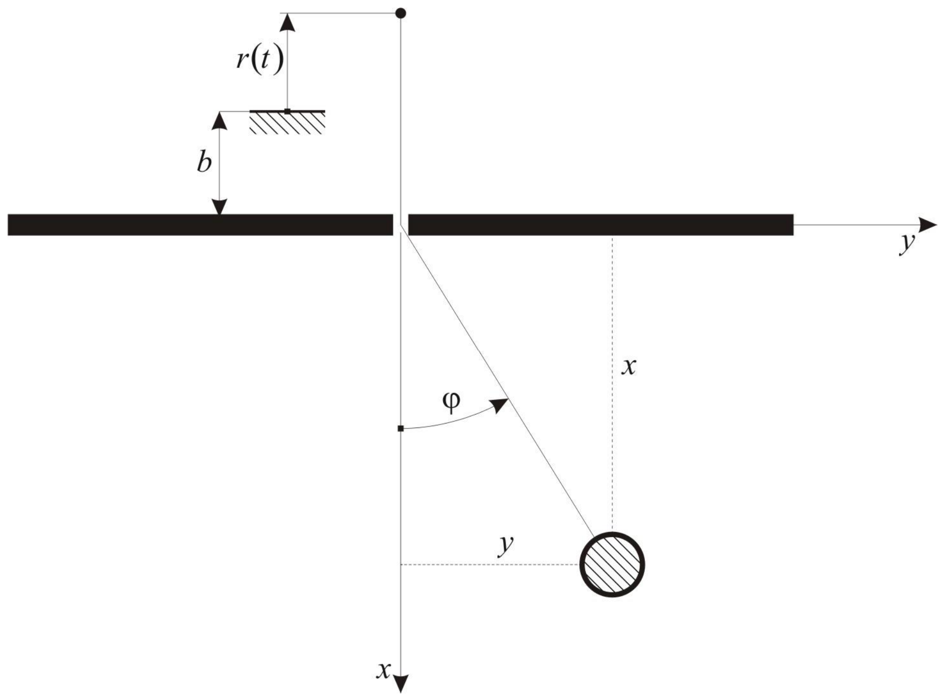

Fig. 4.4

3) The third example of system is point moving along the Ădog curveñ see Fig. 4.5 whose tangent is directed in each time instant into the point *B*. This point moves along the straight line $x = l_0$ by prescribed motion. The name Ădogĝs curveñ corresponds to behaviour of the dog watching a biker going on the road. Point has 2 DOF in a plane but its trajectory is described by 3 coordinates $dx, dy, r(t)$ (two of them are given by increments). The coupling condition follows from homothety of triangles

$$\frac{dy}{dx} = \frac{r(t) - y}{l_0 - x} \Rightarrow (l_0 - x)dy - [r(t) - y]dx = 0. \tag{4.6}$$

As follows from coupling condition (4.6) it condition can not be written down in the integral form because the exprssion on the left hand side does not represent total differential and for this reason the left hand side can not be integrated.

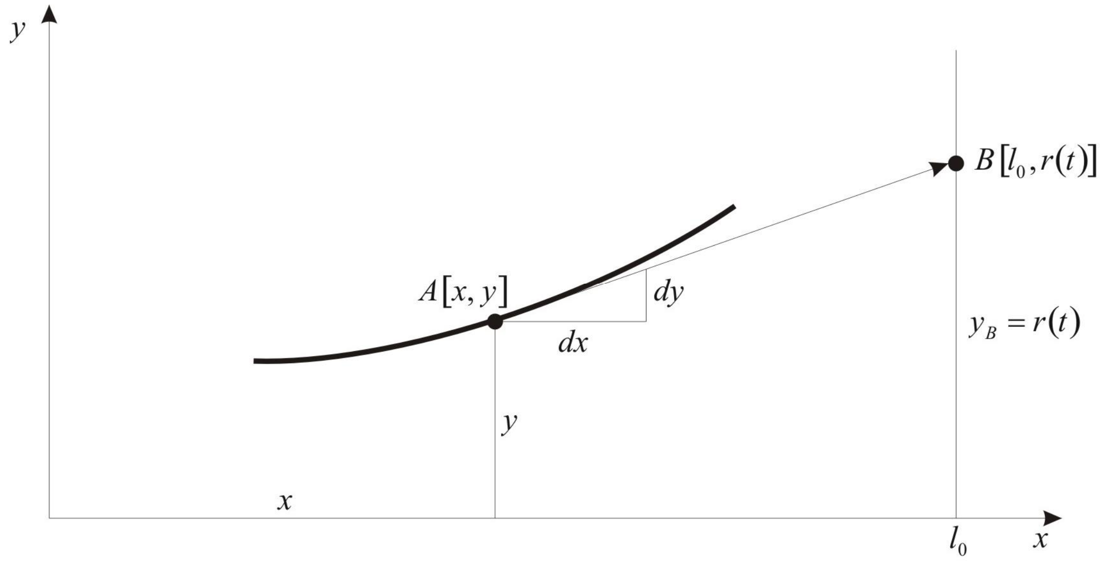

Fig. 4.5

Partial conclusion: All coupling equations (couplings) written down in differential form can be expressed in so called Pfaffôs form (for one coupling equation)

$$\sum_{i=1}^{m} a_i dx_i + a_0 dt = 0, \tag{4.7}$$

where *m* is again number of generalized coordinates determining system position. Generally, the system can contain $r = m - n \ (r < m)$ coupling equations expressed in Pfaffôs form

$$\sum_{i=1}^{m} a_{ij} dx_i + a_{0j} dt = 0, \quad j = 1, 2, \dots, r, \tag{4.8}$$

which can be written down in matrix form

$$\mathbf{A}^T(\mathbf{x}, t) d\mathbf{x} + \mathbf{a}_0(\mathbf{x}, t) dt = 0. \tag{4.9}$$

As one can see in (4.9) the coefficients staying at coordinate and time differentials are functions of corresponding coordinates and time

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbf{R}^{m,1}, \quad d\mathbf{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} \in \mathbf{R}^{m,1}, \ \mathbf{A}^T(\mathbf{x}, t) \in \mathbf{R}^{j,m}, \ \mathbf{a}_0(\mathbf{x}, t) \in \mathbf{R}^{r,1}. \tag{4.10}$$

Now we can come to the declared classification of the mechanical discrete systems.

a) When the couplings are integrable the equation (4.9) can be rewritten into form

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{0}. \tag{4.11}$$

Then the couplings and whole system is said to be **holonomic**. Turning back to the three presented systems we can classify the systems 1) and 2) as holonomic. Its couplings can be expressed in integral and differential form, respectively. From the other side the system 3) (Ădogôs curveñ) is non-holonomic because the coupling condition exists only in differential form.

b) The next criterion of classification is stationarity. If the couplings do not include explicitly time, it means that

$$\mathbf{a}_0 = \mathbf{0}\,, \tag{4.12}$$

in (4.9), the couplings and whole system is said to be **stationary (time independent).** According to this criterion the system 1) is stationary but the next other two systems are **reonomic (non-stationary, time dependent).**

Let us suposse for this moment that the investigated system is stationary. Let us arrange the generalized coordinates in order of n independent and subsequently r dependent ones and let us subdivide the $\mathbf{A}^T(\mathbf{x},t)$ matrix into corresponding blocks (sub-matrices) in form

$$\mathbf{A}^T(\mathbf{x},t) = \left[\mathbf{A}_1^T(\mathbf{x},t),\ \mathbf{A}_2^T(\mathbf{x},t)\right],\ \mathbf{x} = \begin{bmatrix}\mathbf{x}_n\\ \mathbf{x}_z\end{bmatrix},\ \ \mathbf{A}_1^T(\mathbf{x},t) \in \mathbf{R}^{r,n},\ \ \mathbf{A}_2^T(\mathbf{x},t) \in \mathbf{R}^{r,r},\ \mathbf{x}_n \in \mathbf{R}^{n,1},\ \mathbf{x}_z \in \mathbf{R}^{r,1}.$$

The equation (4.9) now can be written down as

$$\left[\mathbf{A}_1^T(\mathbf{x},t),\ \mathbf{A}_2^T(\mathbf{x},t)\right]\begin{bmatrix}d\mathbf{x}_n\\ d\mathbf{x}_z\end{bmatrix} = \mathbf{0}, \tag{4.13}$$

and layed down by means of blocks

$$\mathbf{A}_1^T(\mathbf{x},t)\,d\mathbf{x}_n + \mathbf{A}_2^T(\mathbf{x},t)\,d\mathbf{x}_z = \mathbf{0}. \tag{4.14}$$

Supossing that the $\mathbf{A}_2^T(\mathbf{x},t)$ matrix is regular we can express the differential increments of dependent coordinates by means of independent coordinates in form

$$d\mathbf{x}_z = -\mathbf{A}_2^{-T}(\mathbf{x},t)\mathbf{A}_1^T(\mathbf{x},t)\,d\mathbf{x}_n. \tag{4.15}$$

This relation will be used for development of Lagrangeôs equations with couplings.

4.2. Principle of virtual displacements (PVD)

This principle is taken into account as axiom comming out from equilibrium conditions. Let us define so called virtual displacement. It can be seen as differential increment of some of m coordinates describing the system position. Supossing validity of m system equilibrium conditions

$$\sum_i \mathbf{F}_i = \mathbf{0}, \tag{4.16}$$

we can write identity

$$\left(\sum_i \mathbf{F}_i\right)\cdot \delta\mathbf{x} = 0, \tag{4.17}$$

where $\delta\mathbf{x}$ is arbitrary vector of dimension enabling to perform inner product of metioned vectors. The last equation can be written down as

$$\left(\sum_i \mathbf{F}_i\right)^T \delta\mathbf{x} = \delta\mathbf{x}^T\left(\sum_i \mathbf{F}_i\right) = 0. \tag{4.18}$$

The vector $\delta\mathbf{x}$ can be seen as a vector of differentially small displacements. Then the resulting inner product represents the work done by all generalized forces acting on the systém. This work is zero as a result of equilibrium (static or dynamic). Let us remind that the system has n DOF, where $n \le m$. If $m > n$ it is necessary to supplement $r = m - n$ of coupling conditions. In case virtual displacements are mutually independent then $m = n,\ r = 0$ and the individual products forming inner product (4.17), (4.18) are zero.

PVD can be expressed verbally in this way:
If the system is in equilibrium (dynamic or static), then the virtual work of all forces acting on the system is equal to zero.

The vector $\delta \mathbf{x}$ marking differential displacements corresponds to real differential displacements $d\mathbf{x}$ for stationary systems. When the system is reonomic (non-stationary) the situation can be different. Fig. 4.6 shows real displacement dx_i which is effected by time change of $r(t)$ and do not correspond to the virtual displacement δx_i. For this reason we will award such virtual displacement change of the system to be this change not effected by time change. Such a change is called isochronic variation.

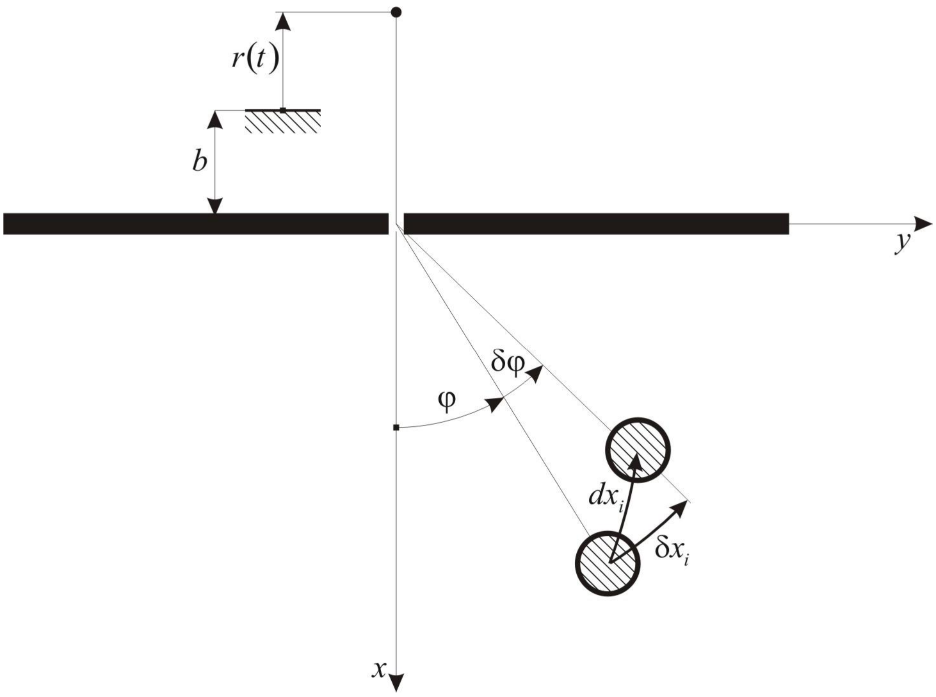

Fig. 4.6

Let us turn back our attention to the note ***Ă****In case virtual displacements are mutually independent then* $m = n,\ r = 0$ *and the individual products forming inner product (4.17), (4.18) are zero****ñ***. Then the relations staying at the individual virtual dispalcements in equation (4.18) have to be zero, it represents $m = n$ equations. These equations correspond to the static equilibrium conditions or equations of motion for static and dynamic problem, respectively. It can be realized in such a way that the system is gradually given by m individual virtual displacements (repeatedly one of the virtual dispalacements will be non-zero while the other displacements will be equal to zero) and the resulting virtual work of those forces will be put equal to zero. The two examples when $m = n$ will be presented as an application to the PVD. Let us remind two rules for modelling:

1) **In mechanics the body (point) position is marked by coordinate oriented from fixed point (polar radius in a case of rotation motion) to the moving point (polar radius)**
2) **The time derivatives of position coordinate and its virtual change have the same direction like this coordinate.**

4.2.1 Application of PVD to the systems described by driving generalized coordinates (*m*=*n*)

Example 4.1

The mechanism with harmonical cam is depicted in Fig. 4.7. Let us assemble its equation of motion. As a starting point we can determine DOF number according to the relation

$$i = 3\times(3-1) - 2(p+r+v) - o = 6 - 2(0+1+0) - 3 = 1. \tag{4.19}$$

When we deside for one system description coordinate, then $m = n = 1$. For this reason we will write only one equation of motion. As independent coordinate we can chose the cam rotation angle φ. The other coordinates and their time derivates including virtual changes have to be expressed by means of φ, $\dot{\varphi}$, $\ddot{\varphi}$ and $\delta\varphi$.

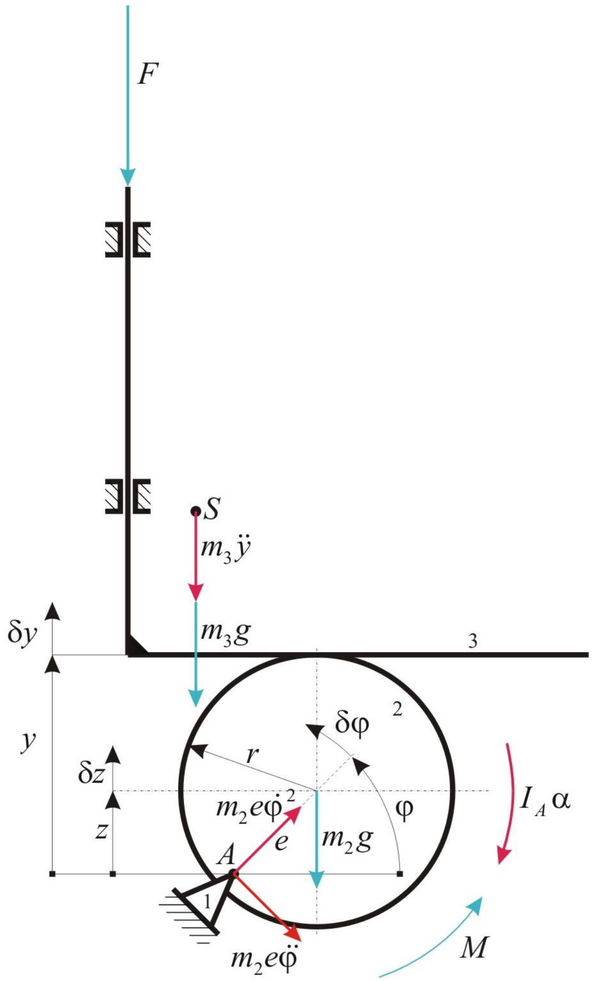

Fig. 4.7

Note:

The main advantage of analytical mechanics methods is fact that the result of its application is directly system of equations of motion. It is not necessary to assemble laboured equilibrium conditions containing also coupling forces. When the couplings are perfect then the coupling forces do not do work and the use of analytical mechanics methods is advantageous. In opossite case (real couplings with passive effects) this advantage disappears because expression of passive effect work can be very difficult.

Let the mentioned system is without passive effects. We fistly give the virtual displacement $\delta\varphi$ to the system. Because of one coordinate decription dependent virtual displacements δy and δz appears. These ones can be computated by the differentiation of relations

$$\begin{aligned} y &= e\sin\varphi + r \Rightarrow \delta y = e\cos\varphi\,\delta\varphi,\\ z &= e\sin\varphi \qquad \Rightarrow \delta z = e\cos\varphi\,\delta\varphi. \end{aligned} \tag{4.20}$$

The time derivative can be obtained in the similar way

$$y = e\sin\varphi + r \Rightarrow \ddot{y} = e\ddot{\varphi}\cos\varphi - e\dot{\varphi}^2\sin\varphi. \tag{4.21}$$

Then the virtual work of all forces acting on the system can be written down in form

$$(M - I_A\ddot{\varphi})\delta\varphi - (F + m_3 g + m_3\ddot{y})\delta y - m_2 g\,\delta z = 0. \tag{4.22}$$

Substituting (4.20) and (4.21) into (4.22) we can come to

$$(M - I_A\ddot{\varphi})\delta\varphi - \left[F + m_3 g + m_3 e\ddot{\varphi}\cos\varphi - m_3 e\dot{\varphi}^2\sin\varphi\right]e\cos\varphi\,\delta\varphi - m_2 g e\cos\varphi\,\delta\varphi = 0. \tag{4.23}$$

The virtual change $\delta\varphi$ is of the differential order but non-zero. For this reason the equation (4.23) can be canceled by $\delta\varphi$ and written as

$$\boxed{M - I_A\ddot{\varphi} - \left[F + m_3 g + m_3 e\ddot{\varphi}\cos\varphi - m_3 e\dot{\varphi}^2\sin\varphi + m_2 g\right]e\cos\varphi = 0,} \tag{4.24}$$

which is the equation of motion of the whole mechanism. The equation of motion assembladge is always starting point for solution of dynamic problem. The question remains what is the goal of solution. If the goal of solution of the equation of motion is motion of mechanism (specific dynamics problem) e.g. running up it is necessary to solve the equation (4.24) with respect to φ. It is strongly non-linear differential equation whose analytical solution we can not find in most cases. Then only one possible way to solution is numerical one. The description of numerical methods for solution of the equation (4.24) extends range of this text book. The reader can be recommended to the reference [5]. The different situation occurs when the kinematic quantity φ or its some time derivative e.g. $\dot{\varphi}(t) = \omega_0$ (kinematic-static problem) is prescribed as a function of time. Then the equation (4.24) becomes the algebraic equation for force quantities e.g. M (supposing thet F is prescribed function of φ or t) for required kinematic function.

Example 4.2

The system with 2 DOF is depicted in Fig. 4.8. To simplify we will suppose that system motion takes place in horizontal plane and mass of the member 3 is neglected. If we decide for solution in space of driving generalized coordinates the result of PVD application will be two equations of motion. As those coordinates will be chosen angles φ and ψ. Then the coordinate y and its time derivatives together with virtual change have to be expressed by means of driving generalized coordinates. It should be sayd that this approach is appropriate esspecially in such cases when dependent coordinates are possible to express as a function of the independent driving generalized coordinates.

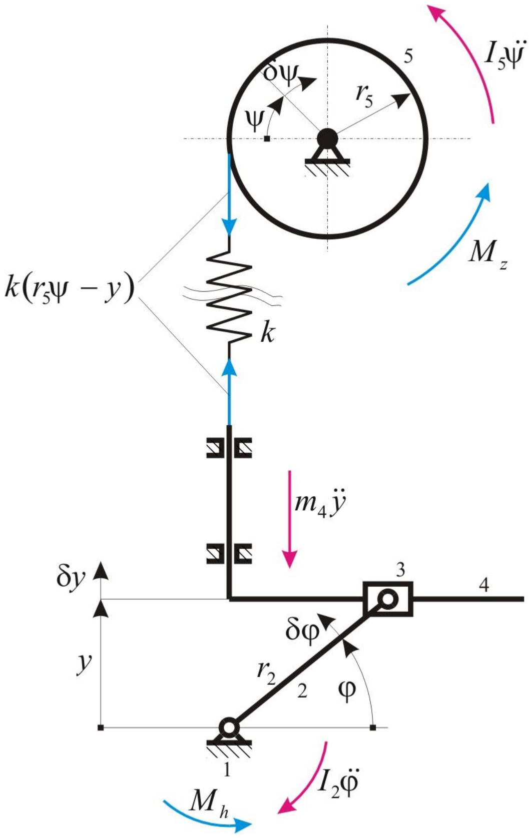

Fig. 4.8

It is realizable in this case by means of derivatives and differentiation of the relation

$$y = r_2 \sin\varphi \;\Rightarrow\; \ddot{y} = r_2\ddot{\varphi}\cos\varphi - r_2\dot{\varphi}^2 \sin\varphi, \qquad \delta y = r_2 \cos\varphi\,\delta\varphi. \tag{4.25}$$

Since the system has 2 DOF and driving coordinates are φ and ψ we assign virtual displacement $\delta\varphi \neq 0,\ \delta\psi = 0$ to the system. Virtual work of the all force effects has to be equal to zero and it can be written as

$$(M_h - I_2\ddot{\varphi})\delta\varphi - m_4\ddot{y}\delta y + k(r_5\psi - y)\delta y = 0. \tag{4.26}$$

Substituting (4.25) into the last equation we can come to

$$(M_h - I_2\ddot{\varphi})\delta\varphi - m_4\left(r_2\ddot{\varphi}\cos\varphi - r_2\dot{\varphi}^2 \sin\varphi\right)r_2\cos\varphi\,\delta\varphi + k(r_5\psi - r_2\sin\varphi)r_2\cos\varphi\,\delta\varphi = 0\,. \tag{4.27}$$

Dividing by $\delta\varphi$ the first equation of motion takes a form

$$\boxed{M_h - I_2\ddot{\varphi} - m_4\left(r_2\ddot{\varphi}\cos\varphi - r_2\dot{\varphi}^2 \sin\varphi\right)r_2\cos\varphi + k(r_5\psi - r_2\sin\varphi)r_2\cos\varphi = 0.} \tag{4.28}$$

Similar approach can be applied for the second driving generalized coordinate. Now we assign virtual displacement $\delta\varphi = 0,\ \delta\psi \neq 0$ to the system. Then the virtual work of load acting on the system will be

$$-(M_z + I_5\ddot{\psi})\delta\psi - k(r_5\psi - y)r_5\delta\psi = 0. \tag{4.29}$$

Substituting y from (4.25) into the last equation and dividing by $-\delta\psi$ we can obtain the second equation of motion in form

$$\boxed{M_z + I_5\ddot{\psi} + kr_5^2\psi - kr_2r_5\sin\varphi = 0.} \tag{4.30}$$

The solution to the equations (4.28) and (4.30) is realizable only in numerical way. Respecting fact that the second derivatives in these equations are linear we can express both equations in form

$$\begin{bmatrix}\ddot{\varphi}\\ \ddot{\psi}\end{bmatrix} = \begin{bmatrix}\dfrac{1}{I_2 + m_4r_2^2\cos^2\varphi}\left[M_h + m_4r_2^2\dot{\varphi}^2\sin\varphi\cos\varphi + k(r_5\psi - r_2\sin\varphi)r_2\cos\varphi\right]\\ \dfrac{1}{I_5}\left(kr_2r_5\sin\varphi - M_z - kr_5^2\psi\right)\end{bmatrix}. \tag{4.31}$$

It can be expressed in matrix way as

$$\ddot{\mathbf{x}} = \breve{\mathbf{g}}(\mathbf{x},\dot{\mathbf{x}}), \tag{4.32}$$

$$\mathbf{x} = \begin{bmatrix}\varphi\\ \psi\end{bmatrix},\quad \breve{\mathbf{g}}(\mathbf{x},\dot{\mathbf{x}}) = \begin{bmatrix}\dfrac{1}{I_2 + m_4r_2^2\cos^2\varphi}\left[M_h + m_4r_2^2\dot{\varphi}^2\sin\varphi\cos\varphi + k(r_5\psi - r_2\sin\varphi)r_2\cos\varphi\right]\\ \dfrac{1}{I_5}\left(kr_2r_5\sin\varphi - M_z - kr_5^2\psi\right)\end{bmatrix}. \tag{4.33}$$

Using some standard procedure for solution of ODE (ordinary differential equation) system it is convenient to transfer the system (4.32) to the double ODE system of the first order. Let us put

$$\mathbf{y}_1 = \dot{\mathbf{x}},\quad \mathbf{y}_2 = \mathbf{x}. \tag{4.34}$$

The the system (4.32) takes form

$$\begin{aligned}\dot{\mathbf{y}}_1 &= \breve{\mathbf{g}}(\mathbf{y}_1,\mathbf{y}_2),\\ \dot{\mathbf{y}}_2 &= \mathbf{y}_1.\end{aligned} \tag{4.35}$$

Two ODE systems (4.35) can be rewritten into compact form

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}), \tag{4.36}$$

$$\text{where}\quad \mathbf{z} = \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix},\quad \mathbf{g}(\mathbf{z}) = \begin{bmatrix}\breve{\mathbf{g}}(\mathbf{y}_1,\mathbf{y}_2)\\ \mathbf{y}_1\end{bmatrix}. \tag{4.37}$$

There exist common procedures for ODE system in standard form (4.36) which are part of SW like e.g. MATLAB or MATHEMATICA

4.2.2 Application of PVD for systems described by generalized coordinates (*m*>*n*)

If the system is described by $m > n$ generalized coordinates there exists Pfaffôs form (4.9) between dependent and driving coordinates. It can be expressed with respect to isochronic variation $\delta t = 0$ in form

$$\mathbf{A}^T \delta\mathbf{x} = \left[\mathbf{A}_1^T(\mathbf{x},t),\ \mathbf{A}_2^T(\mathbf{x},t)\right]\begin{bmatrix}\delta\mathbf{x}_n \\ \delta\mathbf{x}_z\end{bmatrix} = \mathbf{0}. \tag{4.38}$$

$$\mathbf{A}^T(\mathbf{x},t) = \left[\mathbf{A}_1^T(\mathbf{x},t),\ \mathbf{A}_2^T(\mathbf{x},t)\right],\ \mathbf{x} = \begin{bmatrix}\mathbf{x}_n \\ \mathbf{x}_z\end{bmatrix},\ \ \mathbf{A}_1^T(\mathbf{x},t) \in \mathbf{R}^{r,n},\ \ \mathbf{A}_2^T(\mathbf{x},t) \in \mathbf{R}^{r,r},\ \mathbf{x}_n \in \mathbf{R}^{n,1},\ \mathbf{x}_z \in \mathbf{R}^{r,1}$$

Let us remind that $\mathbf{x}_n \in \mathbf{R}^n$ are driving generalized coordinates while $\mathbf{x}_z \in \mathbf{R}^r$ $(r = m - n)$ are dependent (redundant) coordinates. The matrix was already defined by relation (4.13).

Note:
The question suggests itself why we should operate with larger number of coordinates m then is the number of driving coordinates n (DOF). Answer: It is sometimes impossible to express dependent coordinates by means of driving ones in explicit way. Then it is not possible to determine derivatives and virtual changes of dependent coordinates by means of driving coordinates and its derivatives and virtual changes.

The relation (4.16) can be written as

$$\mathbf{F}_a + \mathbf{D} + \mathbf{F}_v = \mathbf{0}, \tag{4.39}$$

where

$$\mathbf{F}_a = \sum_i \mathbf{F}_{ai},\quad \text{resp.}\quad \mathbf{D} = \sum_i \mathbf{D}_i,\quad \text{resp.}\quad \mathbf{F}_v = \sum_i \mathbf{F}_{vi} \tag{4.40}$$

are resulting action and inertial and coupling generalized forces, respectively, acting in directions of individual *m* generalized coordinates. Firstly we can write the relation for virtual work of all forces (4.39)

$$\delta\mathbf{x}^T(\mathbf{F}_a + \mathbf{D} + \mathbf{F}_v) = 0. \tag{4.41}$$

Virtual work of coupling forces acting in perfect couples is zero it means that $\delta\mathbf{x}^T\mathbf{F}_v = 0$. Now the relation (4.38) can be rearranged into form (the equation (4.38) is transposed)

$$\delta\mathbf{x}^T\mathbf{A} = \mathbf{0}. \tag{4.42}$$

The equation (4.42) represents *r* equations on row. To be these equations independent of each other its linear combination equating to zero must exist. It means there exists a vector of linear combination $\boldsymbol{\lambda}$ satisfying condition

$$\delta\mathbf{x}^T\mathbf{A}\boldsymbol{\lambda} = 0. \tag{4.43}$$

Adding both equations (4.41), (4.43) and respecting $\delta\mathbf{x}^T\mathbf{F}_v = 0$, we can come to

$$\boxed{\delta\mathbf{x}^T(\mathbf{F}_a + \mathbf{D} + \mathbf{A}\boldsymbol{\lambda}) = 0.} \tag{4.44}$$

In this moment each of equations obtained in system (4.44) must be satisfied. It means that

$$\mathbf{F}_a + \mathbf{D} + \mathbf{A}\boldsymbol{\lambda} = \mathbf{0}. \tag{4.45}$$

The last relation represents m differential equations for $m+r$ unknowns $\mathbf{x} \in \mathbf{R}^m$, $\boldsymbol{\lambda} \in \mathbf{R}^r$. Now we add to the system (4.45) coupling condition (4.38)

$$\mathbf{A}^T \delta\mathbf{x} = \mathbf{0} \tag{4.46}$$

divided by δt. This condition gets a form

$$\mathbf{A}^T \dot{\mathbf{x}} = \mathbf{0}. \tag{4.47}$$

Derivating the last equation with respect to time we can come to

$$\boxed{\mathbf{A}^T \ddot{\mathbf{x}} = -\dot{\mathbf{A}}^T \dot{\mathbf{x}}.} \tag{4.48}$$

Let us subdivide the action forces into external (independent) forces and forces depending on generalized coordinates and time according to relation

$$\mathbf{F}_a = \mathbf{f}_e(t) - \mathbf{f}_N(\dot{\mathbf{x}}, \mathbf{x}, t). \tag{4.49}$$

Further we can express inertia forces which are linear function of generalized accelerations $\ddot{\mathbf{x}}$ and non-linear function of generalized displacements and velocities $\mathbf{x}$ and $\dot{\mathbf{x}}$ in form

$$\mathbf{D} = -\mathbf{M}\ddot{\mathbf{x}} - \mathbf{f}_D(\dot{\mathbf{x}}, \mathbf{x}). \tag{4.50}$$

Repscting (4.49) and (4.50) we can rewrite the equation (4.45) as

$$\mathbf{M}\ddot{\mathbf{x}} - \mathbf{A}\boldsymbol{\lambda} = \mathbf{f}_e(t) - \mathbf{f}_N(\dot{\mathbf{x}}, \mathbf{x}, t) - \mathbf{f}_D(\dot{\mathbf{x}}, \mathbf{x}). \tag{4.51}$$

This equation together with (4.48) can be rearranged into compact form

$$\boxed{\begin{bmatrix} \mathbf{M}, & \mathbf{A}(\mathbf{x}) \\ \mathbf{A}^T(\mathbf{x}), & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e(t) - \mathbf{f}_N(\dot{\mathbf{x}}, \mathbf{x}, t) - \mathbf{f}_D(\dot{\mathbf{x}}, \mathbf{x}) \\ -\dot{\mathbf{A}}^T \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{a}(\mathbf{x}, \dot{\mathbf{x}}, t) \\ \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}) \end{bmatrix}.} \tag{4.52}$$

The last equation represents system of algebraic-differential equations. Some possibility of its solution will be shown later.

Note:

Practical realization of this methodology will be performed in this way:

a) We assign always only one non-zero virtual displacement to the system while other displacements will be equal to zero. After dividing by this virtual displacement we obtain m equations.

b) Then we add system (4.48) and obtain the whole system of $m+r$ *algebraic-differential equations*

Example 4.3

Let us assemble equation of motion os the system see Fig. 4.8 in such a way that number of generalized coordinates will be $m = 3$, (φ, ψ, y). Vector of generalized coordinates and their virtual increments will be written in form

$$\mathbf{x} = \begin{bmatrix} \varphi \\ \psi \\ y \end{bmatrix}, \quad \delta\mathbf{x} = \begin{bmatrix} \delta\varphi \\ \delta\psi \\ \delta y \end{bmatrix}. \tag{4.53}$$

Because number of DOF is $n=2$ it is necessary to supplement $r=m-n=1$ coupling condition

$$y-r_2\sin\varphi=0. \tag{4.54}$$

Pfaffôs form can be obtained by differentiation of coupling condition (4.54). Then we can come to

$$dy-r_2\cos\varphi\,\delta\varphi=0. \tag{4.55}$$

The last relation can be rewritten into matrix form

$$\mathbf{A}^T\delta\mathbf{x}=[-r_2\cos\varphi,\quad 0,\quad 1]\begin{bmatrix}\delta\varphi\\ \delta\psi\\ \delta y\end{bmatrix}\Rightarrow\mathbf{A}=\begin{bmatrix}-r_2\cos\varphi\\ 0\\ 1\end{bmatrix}. \tag{4.56}$$

Derivating matrix **A** with respect to time we obtain

$$\dot{\mathbf{A}}=\begin{bmatrix}r_2\dot\varphi\sin\varphi\\ 0\\ 0\end{bmatrix}. \tag{4.57}$$

Substituting (4.57) and (4.53) into (4.44) we achieve

$$[\delta\varphi,\quad\delta\psi,\quad\delta y]\left\{\begin{bmatrix}M_h-I_2\ddot\varphi\\ -M_z-I_5\ddot\psi-kr_5(r_5\psi-y)\\ k(r_5\psi-y)-m_4\ddot y\end{bmatrix}-\begin{bmatrix}-r_2\cos\varphi\\ 0\\ 1\end{bmatrix}\lambda\right\}=0. \tag{4.58}$$

Therms staying at $\delta\varphi,\ \delta\psi,\ \delta y$ are independent and thus they have to be zero. Then three equations follow from (4.58)

$$\begin{aligned}&M_h-I_2\ddot\varphi+\lambda r_2\cos\varphi=0,\\ &-M_z-I_5\ddot\psi-kr_5(r_5\psi-y)=0,\\ &k(r_5\psi-y)-m_4\ddot y-\lambda=0.\end{aligned} \tag{4.59}$$

Now we add equation (4.48) and come to

$$[-r_2\cos\varphi,\quad 0,\quad 1]\begin{bmatrix}\ddot\varphi\\ \ddot\psi\\ \ddot y\end{bmatrix}=-[r_2\sin\varphi\quad 0\quad 0]\begin{bmatrix}\dot\varphi\\ \dot\psi\\ \dot y\end{bmatrix}. \tag{4.60}$$

These four equations can be written in matrix form

$$\begin{bmatrix}I_2, & 0, & 0, & -r_2\cos\varphi\\ 0, & I_5, & 0, & 0\\ 0, & 0, & m_4, & 1\\ -r_2\cos\varphi, & 0, & 1, & 0\end{bmatrix}\begin{bmatrix}\ddot\varphi\\ \ddot\psi\\ \ddot y\\ -\lambda\end{bmatrix}=\begin{bmatrix}M_h\\ -M_z-kr_5(r_5\psi-y)\\ k(r_5\psi-y)\\ -r_2\dot\varphi^2\sin\varphi\end{bmatrix}. \tag{4.61}$$

Correctness of the last equation will be controlled bellow by application of Lagrangeôs equations with multipliers.

4.3. Hamiltonôs principle

Development of Hamiltonôs principle can be shown on the mass point system. We can start from PVD in form (4.41) which contains only forces doing work

$$\delta\mathbf{x}^T(\mathbf{F}_a+\mathbf{D})=0. \tag{4.62}$$

In this moment we leave out subscript ña*a*ò. New subscript will mark out number of mass point so that virtual work of all forces acting on the system can be written as

$$\begin{aligned}\sum_i \delta\mathbf{x}_i^T(\mathbf{F}_i-m_i\ddot{\mathbf{x}}_i) &= \underbrace{\sum_i \delta\mathbf{x}_i^T\mathbf{F}_i}_{\delta W}-\sum_i \delta\mathbf{x}_i^T m_i\ddot{\mathbf{x}}_i=\delta W-\sum_i m_i\left[\frac{d}{dt}\left(\delta\mathbf{x}_i^T\dot{\mathbf{x}}_i\right)-\delta\dot{\mathbf{x}}_i^T\dot{\mathbf{x}}_i\right]= \\ &=\delta W-\sum_i m_i\frac{d}{dt}\left(\delta\mathbf{x}_i^T\dot{\mathbf{x}}_i\right)+\delta\underbrace{\left\{\sum_i\frac{1}{2}m_i\dot{\mathbf{x}}_i^T\dot{\mathbf{x}}_i\right\}}_{E_k}=0,\end{aligned} \tag{4.63}$$

where kde $\mathbf{x}_i$ represents radius-vector and $\delta\mathbf{x}_i$ is corresponding virtual increment. Correctness of the last relation can be proved by reversed process in such a way that we will differentiate it

$$\delta\left\{\sum_i\frac{1}{2}m_i\dot{\mathbf{x}}_i^T\dot{\mathbf{x}}_i\right\}=\sum_i\frac{1}{2}m_i\left(\delta\dot{\mathbf{x}}_i^T\dot{\mathbf{x}}_i+\dot{\mathbf{x}}_i^T\delta\dot{\mathbf{x}}_i\right)=\sum_i m_i\delta\dot{\mathbf{x}}_i^T\dot{\mathbf{x}}_i. \tag{4.64}$$

Let us remind that both terms in brackets are scalar quantities so that their values are unchangeable under transposition. The equation (4.63) can be then rewritten into form

$$\delta W+\delta E_k=\sum_i m_i\frac{d}{dt}\left(\delta\mathbf{x}_i^T\dot{\mathbf{x}}_i\right). \tag{4.65}$$

Let us perform the integration from t_1 to t_2 and obtain

$$\int_{t_1}^{t_2}\delta(E_k+W)\big|_{\mathbf{x}(t)}dt=\sum_i m_i\int_{t_1}^{t_2}\frac{d}{dt}\left(\delta\mathbf{x}_i^T\dot{\mathbf{x}}_i\right)dt. \tag{4.66}$$

Hamiltonôs principle:

Choosing two fixed points of trajectory $x_{i,initial}$ and $x_{i,final}$ *in fixed instants* t_1 *and* t_2 *, see Fig. 4.9, then so called action achieves an extrem when doing real motion.*

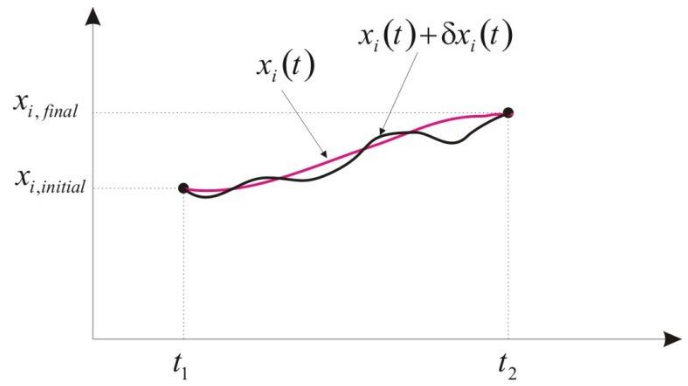

Fig. 4.9

The action is represented by the integral (Hamiltonôs functional)

$$S = \int_{t_1}^{t_2} (E_k + W)\big|_{\mathbf{x}(t)} dt \to extrem. \tag{4.67}$$

This statement cen be easily proved when we express the integral on the right hand side of (4.66) whose value is zero,

$$\sum_i m_i \int_{t_1}^{t_2} \frac{d}{dt}\left(\delta\mathbf{x}_i^T \dot{\mathbf{x}}_i\right) dt = \sum_i m_i \left[\delta\mathbf{x}_i^T \dot{\mathbf{x}}_i\right]_{t_1}^{t_2} = 0, \tag{4.68}$$

because the variations $\delta\mathbf{x}_i^T$ vanish in the boundary points. For this reason even left hand side of (4.66) will be zero. It means that

$$\delta S = \delta \int_{t_1}^{t_2} (E_k + W)\big|_{\mathbf{x}(t)} dt = 0. \tag{4.69}$$

If the variation is zero the action gains an extrem (minimum-without proof). For this reason we can say that the real trajectory minimizes integral (4.67). Own Hamiltonôs principle has small practical meaning but serves to development of Lagrangeôs equations.

4.4 Lagrangeôs equations of the second kind

Let us briefly rewrite

$$\delta W = \sum_i \delta\mathbf{x}_i^T \mathbf{F}_i = \delta\mathbf{x}^T \mathbf{f}_e = \delta^* W, \tag{4.70}$$

where $$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{x}_1 \\ \delta\mathbf{x}_2 \\ \vdots \\ \delta\mathbf{x}_m \end{bmatrix}, \quad \mathbf{f}_e = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_m \end{bmatrix}. \tag{4.71}$$

The star merks fact that the variation is refered only to the trajectory but not to the force. Respecting (4.70) we can then rewrite (4.69) to the form

$$\underbrace{\int_{t_1}^{t_2} \delta E_k dt}_{I} + \int_{t_1}^{t_2} \delta\mathbf{x}^T \mathbf{f}_e dt = 0, \tag{4.72}$$

Integral I can be rearranged (following variation is isochronic-it means that time understood to be constant)

$$\delta \int_{t_1}^{t_2} E_k(\mathbf{x}, \dot{\mathbf{x}}, t) dt = \int_{t_1}^{t_2} \left(\delta\mathbf{x}^T \frac{\partial E_k}{\partial \mathbf{x}} + \delta\dot{\mathbf{x}}^T \frac{\partial E_k}{\partial \dot{\mathbf{x}}} \right) dt = \int_{t_1}^{t_2} \left[\delta\mathbf{x}^T \frac{\partial E_k}{\partial \mathbf{x}} - \delta\mathbf{x}^T \frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}} \right) \right] dt + \underbrace{\left[\delta\mathbf{x}^T \frac{\partial E_k}{\partial \dot{\mathbf{x}}} \right]_{t_1}^{t_2}}_{0}. \tag{4.73}$$

Integrating we have used the per-partes method. Substituting (4.73) into (4.69) we can come to

$$\delta\mathbf{x}^T \int_{t_1}^{t_2} \left[\frac{\partial E_k}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}} \right) + \mathbf{f}_e \right] dt = 0. \tag{4.74}$$

The last relation expresses inner product of two vectors forming summ of products of independent quantities (individual components of vector $\delta\mathbf{x}$), so that each of these products have to be zero. Its consequence is fact that each square bracket in (4.74) has to be equal to zero

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)-\frac{\partial E_k}{\partial \mathbf{x}}=\mathbf{f}_e. \tag{4.75}$$

This is a system of Lagrangeôs equations of the second order expressed in a matrix form. Individual equations can be written as

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{x}_j}\right)-\frac{\partial E_k}{\partial x_j}=F_j, \qquad j=1,\,2,\,\ldots,\,n, \tag{4.76}$$

where n is number of DOF. The right hand side contains external forces while inertia forces are represented by kinetic energy.

4.5 Lagrangeôs equations of mixed type (LRMT)

The use of LRMT described by $m>n$ generalized coordinates (n is number of DOF) for assembladge of equations of motion is advantageous esspecially in the case when redundant coordinates can not be explicitly expressed by means of driving coordinates. Then we must add $r=m-n$ coupling conditions

$$f_j(\mathbf{x})=0, \quad j=1,\,2,\ldots,\,r, \tag{4.77}$$

to those m equations describing dynamic equilibrium conditions. The last relation can be written in matrix form

$$\mathbf{f}(\mathbf{x})=\mathbf{0}, \quad \mathbf{f}(\mathbf{x})\in\mathbf{R}^r. \tag{4.78}$$

It is about the problem of bound extreme. The Hamiltonôs principle will be extended by the corresponding vector of Lagrangeôs multipliers

$$S=\int_{t_1}^{t_2}\left[E_k+W+\mathbf{f}^T(\mathbf{x})\boldsymbol{\lambda}\right]_{\mathbf{x}(t)}dt \rightarrow \textit{extreme.} \tag{4.79}$$

To calculate a functional variation let us differentiate the equation (4.79)

$$\delta S=\delta\int_{t_1}^{t_2}\left(E_k+W+\delta\mathbf{f}^T(\mathbf{x})\boldsymbol{\lambda}\right)\Big|_{\mathbf{x}(t)}dt=0, \tag{4.80}$$

where

$$\delta\mathbf{f}^T(\mathbf{x})=\delta\mathbf{x}^T\,\frac{\partial\mathbf{f}^T(\mathbf{x})}{\partial\mathbf{x}} \tag{4.81}$$

This relation is coupling condition in differential form which is represented by Pfaffôs form (4.42). Comparing (4.81) and (4.42) we can come to the fact that the matrix of Pfaffôs form takes form

$$\mathbf{A}(\mathbf{x})=\frac{\partial\mathbf{f}^T(\mathbf{x})}{\partial\mathbf{x}}\in\mathbf{R}^{r,m}. \tag{4.82}$$

Variation of kinetic energy and work expressed by relation (4.74) after substituting into (4.80) brings an equation

$$\delta\mathbf{x}^T\int_{t_1}^{t_2}\left[\frac{\partial E_k}{\partial\mathbf{x}}-\frac{d}{dt}\left(\frac{\partial E_k}{\partial\dot{\mathbf{x}}}\right)+\mathbf{f}_e+\frac{\partial\mathbf{f}^T(\mathbf{x})}{\partial\mathbf{x}}\boldsymbol{\lambda}\right]dt=0. \tag{4.83}$$

Integral (4.83) will be zero when its integrand will be zero too, it means

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)-\frac{\partial E_k}{\partial \mathbf{x}}=\mathbf{f}_e+\frac{\partial \mathbf{f}^T(\mathbf{x})}{\partial \mathbf{x}}\boldsymbol{\lambda}, \tag{4.84}$$

which respecting (4.82) can be rewritten in form

$$\boxed{\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)-\frac{\partial E_k}{\partial \mathbf{x}}=\mathbf{f}_e+\mathbf{A}(\mathbf{x})\boldsymbol{\lambda}.} \tag{4.84a}$$

This equation is itemized as

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{x}_i}\right)-\frac{\partial E_k}{\partial x_i}=F_i+\sum_{j=1}^{r}\frac{\partial f_j(\mathbf{x})}{\partial x_i}\lambda_j, \qquad i=1,\, 2,\, \dots,\, m=n+r. \tag{4.84b}$$

Equations (4.84) represent $m=n+r$ algebraic-differential equations for unknowns $\mathbf{x}(t)\in\mathbf{R}^{m,1}$ and vector of Lagrange's multipliers $\boldsymbol{\lambda}\in\mathbf{R}^{r,1}$. The system will be supplemented by coupling condition (4.78)

$$\mathbf{f}(\mathbf{x})=\mathbf{0}, \quad \mathbf{f}(\mathbf{x})\in\mathbf{R}^r. \tag{4.85}$$

Both equations (4.84) and (4.85) present system of $m+r=n+2r$ algebraic-differential equations for $m+r$ unknowns $\mathbf{x}(t)\in\mathbf{R}^{m,1}$ and $\boldsymbol{\lambda}\in\mathbf{R}^{r,1}$. From practical reasons it is suitable to twice differentiate the equation (4.85)

$$\dot{\mathbf{f}}(\mathbf{x})=\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\dot{\mathbf{x}}=\mathbf{A}^T(\mathbf{x})\dot{\mathbf{x}}=\mathbf{0}, \tag{4.86}$$

$$\boxed{\ddot{\mathbf{f}}(\mathbf{x})=\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}}+\dot{\mathbf{A}}^T(\mathbf{x})\dot{\mathbf{x}}=\mathbf{A}^T(\mathbf{x})\ddot{\mathbf{x}}+\dot{\mathbf{A}}^T(\mathbf{x})\dot{\mathbf{x}}=\mathbf{0}.} \tag{4.87}$$

Now we rearrange the relations staying on the left hand side of (4.84). Derivatives of kinetic energy $E_k=E_k(\mathbf{x},\dot{\mathbf{x}})$ can be expressed in form

$$\frac{\partial E_k}{\partial \dot{\mathbf{x}}}=\begin{bmatrix}\dfrac{\partial E_k}{\partial \dot{x}_1}\\ \dfrac{\partial E_k}{\partial \dot{x}_2}\\ \vdots\\ \dfrac{\partial E_k}{\partial \dot{x}_m}\end{bmatrix}, \quad \frac{\partial E_k}{\partial \mathbf{x}}=\begin{bmatrix}\dfrac{\partial E_k}{\partial x_1}\\ \dfrac{\partial E_k}{\partial x_2}\\ \vdots\\ \dfrac{\partial E_k}{\partial x_m}\end{bmatrix}, \tag{4.88}$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)=\frac{\partial}{\partial \mathbf{x}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)\dot{\mathbf{x}}+\frac{\partial}{\partial \dot{\mathbf{x}}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)\ddot{\mathbf{x}}=\frac{\partial}{\partial \mathbf{x}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)\dot{\mathbf{x}}+\mathbf{M}(\mathbf{x})\ddot{\mathbf{x}}, \tag{4.89}$$

where

$$\frac{\partial}{\partial \mathbf{x}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)=\begin{bmatrix} \dfrac{\partial^2 E_k}{\partial \dot{x}_1 \partial x_1}, & \dfrac{\partial^2 E_k}{\partial \dot{x}_1 \partial x_2}, & \ldots, & \dfrac{\partial^2 E_k}{\partial \dot{x}_1 \partial x_m} \\ \dfrac{\partial^2 E_k}{\partial \dot{x}_2 \partial x_1}, & \dfrac{\partial^2 E_k}{\partial \dot{x}_2 \partial x_2}, & \ldots, & \dfrac{\partial^2 E_k}{\partial \dot{x}_2 \partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 E_k}{\partial \dot{x}_m \partial x_1}, & \dfrac{\partial^2 E_k}{\partial \dot{x}_m \partial x_2}, & \ldots, & \dfrac{\partial^2 E_k}{\partial \dot{x}_m \partial x_m} \end{bmatrix} \in \mathbf{R}^{m,m}, \tag{4.90}$$

$$\mathbf{M}(\mathbf{x})=\frac{\partial}{\partial \dot{\mathbf{x}}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)=\begin{bmatrix} \dfrac{\partial^2 E_k}{\partial \dot{x}_1 \partial \dot{x}_1}, & \dfrac{\partial^2 E_k}{\partial \dot{x}_1 \partial \dot{x}_2}, & \ldots, & \dfrac{\partial^2 E_k}{\partial \dot{x}_1 \partial \dot{x}_m} \\ \dfrac{\partial^2 E_k}{\partial \dot{x}_2 \partial \dot{x}_1}, & \dfrac{\partial^2 E_k}{\partial \dot{x}_2 \partial \dot{x}_2}, & \ldots, & \dfrac{\partial^2 E_k}{\partial \dot{x}_2 \partial \dot{x}_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 E_k}{\partial \dot{x}_m \partial \dot{x}_1}, & \dfrac{\partial^2 E_k}{\partial \dot{x}_m \partial \dot{x}_2}, & \ldots, & \dfrac{\partial^2 E_k}{\partial \dot{x}_m \partial \dot{x}_m} \end{bmatrix} \in \mathbf{R}^{m,m}. \tag{4.91}$$

Now we rearrange (4.84a) and (4.87) into form

$$\mathbf{M}(\mathbf{x})\ddot{\mathbf{x}}-\mathbf{A}(\mathbf{x})\boldsymbol{\lambda}=\mathbf{f}_e-\frac{\partial}{\partial \mathbf{x}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)\dot{\mathbf{x}}, \tag{4.92}$$

$$\mathbf{A}^T(\mathbf{x})\ddot{\mathbf{x}}=-\dot{\mathbf{A}}^T(\mathbf{x})\dot{\mathbf{x}}. \tag{4.93}$$

Let us mark

$$\mathbf{a}(\mathbf{x},\dot{\mathbf{x}},t)=\mathbf{f}_e-\frac{\partial}{\partial \mathbf{x}^T}\left(\frac{\partial E_k}{\partial \dot{\mathbf{x}}}\right)\dot{\mathbf{x}}, \quad \mathbf{b}(\mathbf{x},\dot{\mathbf{x}})=-\dot{\mathbf{A}}^T(\mathbf{x})\dot{\mathbf{x}} \tag{4.94}$$

and both equatins (4.92) and (4.93) take a form

$$\boxed{\begin{bmatrix} \mathbf{M}, & \mathbf{A}(\mathbf{x}) \\ \mathbf{A}^T(\mathbf{x}), & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{x}} \\ -\boldsymbol{\lambda} \end{bmatrix}=\begin{bmatrix} \mathbf{a}(\mathbf{x},\dot{\mathbf{x}},t) \\ \mathbf{b}(\mathbf{x},\dot{\mathbf{x}}) \end{bmatrix},} \tag{4.95}$$

which is identical equation with (4.52) obtained by the PVD application.

4.6 Application of Lagrangeõs equations of the second kind and mixed type

To show the connection between virtual displacement principle and Lagrangeõs equations of the second kind let us solve the examples 4.1, 4.2 and 4.3 by means of Lagrangeõs equations

Example 4.4

The mechanism with harmonic cam is depicted in Fig. 4.7. Let us assemble the equation of motion by means of Lagrangeõs equations of the second kind using description by driving coordinates. This mechanism has 1 DOF and the cam angle was chosen as driving generalized coordinate. The coordinates z, y and its derivatives and variations (differential increments) should be expressed explicitly by means of angle ű and its variation

$$\begin{aligned} y &= e\sin\varphi + r \Rightarrow \delta y = e\cos\varphi\,\delta\varphi, \quad \dot{y} = e\dot{\varphi}\cos\varphi \\ z &= e\sin\varphi \qquad \Rightarrow \delta z = e\cos\varphi\,\delta\varphi, \quad \dot{z} = e\dot{\varphi}\cos\varphi. \end{aligned} \tag{4.96}$$

Kinetic energy can be written in form

$$E_k = \frac{1}{2} I_A \dot{\varphi}^2 + \frac{1}{2} m_3 \dot{y}^2 = \frac{1}{2} I_A \dot{\varphi}^2 + \frac{1}{2} m_3 e^2 \dot{\varphi}^2 \cos^2 \varphi. \tag{4.97}$$

$$\frac{\partial E_k}{\partial \dot{\varphi}} = I_A \dot{\varphi} + m_3 e^2 \dot{\varphi} \cos^2 \varphi, \quad \frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\varphi}}\right) = I_A \ddot{\varphi} + m_3 e^2 \ddot{\varphi} \cos^2 \varphi - 2 m_3 e^2 \dot{\varphi}^2 \sin \varphi \cos \varphi, \tag{4.98}$$

$$\frac{\partial E_k}{\partial \varphi} = -m_3 e^2 \dot{\varphi}^2 \sin \varphi \cos \varphi. \tag{4.99}$$

Generalized force can be determined in the same way like in virual displacement application. It means to compare all force quantity (except inertial) virtual work with the virtual work of generalized force. Taking (4.76) into account we can write

$$F_\varphi \delta\varphi = M \delta\varphi - (F + m_3 g)\delta y - m_2 g \delta z, \tag{4.100}$$

And respecting (4.96) we can come to

$$F_\varphi \delta\varphi = M \delta\varphi - (F + m_3 g) e \cos\varphi \, \delta\varphi - m_2 g e \cos\varphi \, \delta\varphi \tag{4.101}$$

and further

$$F_\varphi = M - (F + m_3 g) e \cos\varphi - m_2 g e \cos\varphi. \tag{4.102}$$

Substituting (4.98), (4.99) and (4.102) into (4.76) $(n = 1)$ we have equation

$$\boxed{M - I_A \ddot{\varphi} - \left[F + m_3 g + m_3 e \ddot{\varphi} \cos\varphi - m_3 e \dot{\varphi}^2 \sin\varphi + m_2 g\right] e \cos\varphi = 0,} \tag{4.103}$$

identical with (4.24).

Example 4.5

The system with 2 DOF is depicted in Fig. 4.8. Let us assemble the equations of motion by means of Lagrangeős equations of the second kind using description by driving generalized coordinates φ and ψ. System position is described by three coordinates φ, ψ and y from which y can be explicitly expressed by means of φ in form

$$y = r_2 \sin\varphi \Rightarrow \delta y = r_2 \cos\varphi \, \delta\varphi, \quad \dot{y} = r_2 \dot{\varphi} \cos\varphi. \tag{4.104}$$

Kinetic energy is written as

$$E_k = \frac{1}{2} I_2 \dot{\varphi}^2 + \frac{1}{2} m_4 \dot{y}^2 + \frac{1}{2} I_5 \dot{\psi}^2. \tag{4.105}$$

Having expressed kinetic energy by means of driving generalized coordinates and respecting (4.104) we can write

$$E_k = \frac{1}{2} I_2 \dot{\varphi}^2 + \frac{1}{2} m_4 r_2^2 \dot{\varphi}^2 \cos^2 \varphi + \frac{1}{2} I_5 \dot{\psi}^2. \tag{4.106}$$

Derivatives needed for substitution into (4.76) have form

$$\frac{\partial E_k}{\partial \dot{\varphi}} = I_2 \dot{\varphi} + m_4 r_2^2 \dot{\varphi} \cos^2 \varphi, \quad \frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\varphi}}\right) = I_2 \ddot{\varphi} + m_4 r_2^2 \ddot{\varphi} \cos^2 \varphi - 2 m_4 r_2^2 \dot{\varphi}^2 \sin\varphi \cos\varphi, \tag{4.107}$$

$$\frac{\partial E_k}{\partial \varphi} = -m_4 r_2^2 \dot{\varphi}^2 \sin\varphi \cos\varphi. \tag{4.108}$$

Now let us give virtual displacement $\delta\varphi \neq 0,\ \ \delta\psi = 0$ to the system. The generalized force F_φ can be obtained by comparison of generalized force work and force effects acting on the system (except inertial effects) corresponding to mentioned virtual displacement

$$F_\varphi \delta\varphi = M_h \delta\varphi + k(r_5\psi - y)\delta y. \tag{4.109}$$

After substitution of (4.104) into (4.109) we obtain

$$F_\varphi \delta\varphi = M_h \delta\varphi + k(r_5\psi - r_2 \sin\varphi) r_2 \cos\varphi\, \delta\varphi \tag{4.110}$$

and after cancelation by $\delta\varphi$ we can come to

$$F_\varphi = M_h + k(r_5\psi - r_2 \sin\varphi) r_2 \cos\varphi. \tag{4.111}$$

Let us substitute (4.107), (4.108) and (4.111) into (4.76) and then we can get the first equation of motion

$$\boxed{I_2\ddot{\varphi} + m_4 r_2^2 \ddot{\varphi} \cos^2\varphi - m_4 r_2^2 \dot{\varphi}^2 \sin\varphi\cos\varphi = M_h + k(r_5\psi - r_2 \sin\varphi) r_2 \cos\varphi,} \tag{4.112}$$

which is identical equation with (4.28). The similar approach will be used for the second driving generalized coordinate.

$$\frac{\partial E_k}{\partial \dot{\psi}} = I_5 \dot{\psi}, \quad \frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\psi}}\right) = I_5 \ddot{\psi}, \tag{4.113}$$

$$\frac{\partial E_k}{\partial \psi} = 0. \tag{4.114}$$

Let us give virtual displacement $\delta\varphi = 0,\ \ \delta\psi \neq 0$ to the system and compare the generalized force F_ψ virtual work with the wirtual work of force effects acting on the system (except inertial effects) corresponding to that virtual displacement

$$F_\psi \delta\psi = -M_z \delta\psi - k(r_5\psi - r_2 \sin\varphi) r_5 \delta\psi. \tag{4.115}$$

Canceling by $\delta\psi$ we can write

$$F_\psi = -M_z - k(r_5\psi - r_2 \sin\varphi) r_5. \tag{4.116}$$

Substituting (4.113), (4.114) a (4.116) into (4.76) we can come to the second equation of motion in form

$$\boxed{I_5\ddot{\psi} = -M_z - k(r_5\psi - r_2 \sin\varphi) r_5,} \tag{4.117}$$

identical with (4.30) which was obtained by application of the virtual displacement principle.

Example 4.6

As the latest demonstration we can show an example similar to 4.3 (Fig. 4.8) presenting equation of motion assembladge of the system whose position is described by three generalized coordinates $m=3,\ (\varphi,\psi,\ y)$. Vector of generalized coordinates and its virtual increments will likewise in example 4.3 have form

$$\mathbf{x}=\begin{bmatrix}\varphi\\ \psi\\ y\end{bmatrix},\quad \delta\mathbf{x}=\begin{bmatrix}\delta\varphi\\ \delta\psi\\ \delta y\end{bmatrix}. \tag{4.118}$$

Because the system has 2 DOF it is necessary to add $r=m-n=1$ coupling condition

$$\mathbf{f}(\mathbf{x})=f(\mathbf{x})=y-r_2\sin\varphi=0. \tag{4.119}$$

Kinetic energy can be expressed by means of those mentioned generalized coordinates in form

$$E_k=\frac{1}{2}I_2\dot{\varphi}^2+\frac{1}{2}m_4\dot{y}^2+\frac{1}{2}I_5\dot{\psi}^2. \tag{4.120}$$

Corresponding derivatives can be written as

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial\dot{\varphi}}\right)=I_2\ddot{\varphi},\ \frac{d}{dt}\left(\frac{\partial E_k}{\partial\dot{\psi}}\right)=I_2\ddot{\psi},\ \frac{d}{dt}\left(\frac{\partial E_k}{\partial\dot{y}}\right)=m_4\ddot{y}, \tag{4.121}$$

$$\frac{\partial E_k}{\partial\varphi}=\frac{\partial E_k}{\partial\psi}=\frac{\partial E_k}{\partial y}=0. \tag{4.122}$$

Now let us give gradually non-zero virtual displacements: Taking $\delta\varphi\neq 0,\ \delta\psi=0,\ \delta y=0$ and comparing virtual works we can come to

$$F_\varphi\delta\varphi=M_h\delta\varphi\Rightarrow F_\varphi=M_h. \tag{4.123}$$

Taking $\delta\varphi=0,\ \delta\psi\neq 0,\ \delta y=0$ we can write

$$F_\psi\delta\psi=-M_z\delta\psi-k(r_5\psi-y)r_5\delta\psi\quad\Rightarrow\quad F_\psi=-M_z-kr_5(r_5\psi-y) \tag{4.124}$$

and for $\delta\varphi=0,\ \delta\psi=0,\ \delta y\neq 0$ the corresponding virtual work and generalized force will take form

$$F_y\delta y=k(r_5\psi-y)\delta y\quad\Rightarrow\quad F_y=k(r_5\psi-y). \tag{4.125}$$

Derivatives of couplings (4.119) with respect to generalized coordinates take vector form

$$\frac{\partial^T\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}=\frac{\partial f(\mathbf{x})}{\partial\mathbf{x}}=\begin{bmatrix}\dfrac{\partial f}{\partial\varphi}\\ \dfrac{\partial f}{\partial\psi}\\ \dfrac{\partial f}{\partial y}\end{bmatrix}=\begin{bmatrix}-r_2\sin\varphi\\ 0\\ 1\end{bmatrix}=\mathbf{A}(\mathbf{x}). \tag{4.126}$$

Now the first three algebraic-differential equations for unknowns $\varphi,\psi,\ y$ and λ can be assembled by substitution (4.126), (4.121-4.125) into (4.84b)

$$\begin{aligned}I_2\ddot{\varphi}&=M_h-\lambda r_2\cos\varphi,\\ I_5\ddot{\psi}&=-M_z-k(r_5\psi-y)r_5,\\ m_4\ddot{y}&=k(r_5\psi-y)+\lambda.\end{aligned} \tag{4.126}$$

These equations can be written in matrix form

$$\mathbf{M}\ddot{\mathbf{x}}(t) = \mathbf{f}_e(t) + \mathbf{A}\boldsymbol{\vartheta}, \tag{4.127}$$

where

$$\mathbf{M} = \begin{bmatrix} I_2 & & \\ & I_5 & \\ & & m_4 \end{bmatrix},\ \mathbf{f}_e(t) = \begin{bmatrix} M_h \\ -M_z - k(r_5\psi - y)r \\ k(r_5\psi - y) \end{bmatrix},\ \mathbf{A}(\mathbf{x}) = \begin{bmatrix} -r_2 \sin\varphi \\ 0 \\ 1 \end{bmatrix},\ \boldsymbol{\vartheta} = \lambda. \tag{4.128}$$

Twice differentiated coupling vector (4.119) has to be added to (4.128), (in this case only one component).

$$\mathbf{f}(\mathbf{x}) = f(\mathbf{x}) = y - r_2 \sin\varphi = 0, \tag{4.129}$$

$$\dot{f}(\mathbf{x}) = \dot{y} - r_2\dot{\varphi}\cos\varphi = 0, \tag{4.130}$$

$$\ddot{f}(\mathbf{x}) = \ddot{y} - r_2\ddot{\varphi}\cos\varphi + r_2\dot{\varphi}^2 \sin\varphi = \mathbf{A}^T(\mathbf{x})\ddot{\mathbf{x}}(t) = 0. \tag{4.131}$$

Adding (4.131) in scalar form to the Eqs. (4.126) we can write down the total system of algebraic-differential equations in form

$$\begin{bmatrix} I_2, & 0, & 0, & -r_2\cos\varphi \\ 0, & I_5, & 0, & 0 \\ 0, & 0, & m_4, & 1 \\ -r_2\cos\varphi, & 0, & 1, & 0 \end{bmatrix} \begin{bmatrix} \ddot{\varphi} \\ \ddot{\psi} \\ \ddot{y} \\ -\lambda \end{bmatrix} = \begin{bmatrix} M_h \\ -kr_5(r_5\psi - y) \\ k(r_5\psi - y) \\ -r_2\dot{\varphi}^2 \sin\varphi \end{bmatrix}, \tag{4.132}$$

which is identic with the system (4.61) obtained by means of PVD. Similarly the matrix form of (4.131) can be added to the Eq. (4.127) and then we can come to

$$\boxed{\begin{bmatrix} \mathbf{M}, & \mathbf{A}(\mathbf{x}) \\ \mathbf{A}^T(\mathbf{x}), & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}} \\ -\boldsymbol{\vartheta} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e(t) \\ -\dot{\mathbf{A}}(\mathbf{x})^T \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{a}(\mathbf{x}) \\ \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}) \end{bmatrix}}, \tag{4.133}$$

which is identic equation with (4.52) where the non-linear vector of general inertia forces $\mathbf{f}_D(\dot{\mathbf{x}}, \mathbf{x})$ and vector of non-linear elastic and viscous forces $\mathbf{f}_N(\dot{\mathbf{x}}, \mathbf{x}, t)$ are zero vectors

5. Vibration of linear discrete systems

We pay our attention in this chapter to weak damped linear vibrating dynamic discrete systems. It means they have finite number of DOF with so called comutative damping matrix (it will be bellow specified). These systems are generally described by system of ordinary differential equations (ODE) of the second order. Moreover we will focuse to systems with constant symmetrical matrices **M**, **B**, **K**. For interested person for vibration of more general systems (continuum, discrete systems with general matrices, stochastic systems) we can recommend reference [5]. Further we supposse symmetrical matrices **M**, **B**, **K** and stability of system. There is whole family of stability definitions of vibrating systems. We restrict our stability definition to verbal expression marking as stable system such group of systems whose displacements stay finite without external excitation. Following this definition we can incorporate into this group also undamped vibrating systems which do not dissipate energy and whose displacements do not decay. Dissipative systems whose displacements are approaching to the equilibrium position without external action can be marked as asymptoticly stable. This chapter will contain only so called initial value problems whose solution always exist.

Note:

Initial value problem of matrix differential equation of r-th order (the highest derivative with respect of time) is understood as problem of solution of this differential equation respecting values of $0, 1, ..., r-1$*-th derivatives of vector of unknowns in one time instant* $t = t_0$ and most frequently we will choose $t_0 = 0$.

5.1 Equation of motion

The equation of motion can be assembled by means of methods of analytical mechanics presented in chapter 4. When the system is discrete and linear its equation of motion can take a form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{B}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t), \tag{5.1}$$

where $\mathbf{M}, \mathbf{B}, \mathbf{K} \in \mathbf{R}^{n,n}$, $\mathbf{f}(t) \in \mathbf{R}^n$ and $\mathbf{q}(t) \in \mathbf{R}^n$. Matrices $\mathbf{M}, \mathbf{B}, \mathbf{K}$ are frequently marked as mass, damping and stiffness matrices, respectively. Symbols $\mathbf{f}(t)$ and $\mathbf{q}(t)$ corresponds to vector of generalized forces and displacements, respectively. When the generalized displacement is rotation angle the corresponding meaning of generalized force is torque. For uniqueness of solution of equation (5.1) it necessary to give the initial conditions (see the last note)

$$\mathbf{q}_0 = \mathbf{q}(0) \in \mathbf{R}^n, \quad \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}(0) \in \mathbf{R}^n. \tag{5.2}$$

The presented solution will use decomposition into vibration mode shapes (eigenvectors) by means of so called modal method. Starting point of mentioned method is determination of mode shapes (eigenvectors) and eigenvalues (modal analysis). The eigenvalues correspond to squares of eigenfrequencies whose units are $[rad/s]$.

5.2 Modal analysis

Modal quantities of weakly damped linear systems (eigenvectors and eigenvalues) are affected neither by input (excitation) nor by damping. Let us remind the relation between eigenfrequency of damped and undamped system [2], respectively

$$\Omega_D = \Omega\sqrt{1-D^2}\,, \tag{5.3}$$

where Ω_D, Ω, D are successively eigenfrequency of damped system, undamped system and damping ratio, respectively. Suppossing that damping ratio of machine structures takes values from interval $D \in \langle 0.01 \div 0.05 \rangle$ difference between eigenfrequancy of undamped and damped system is $\Delta\Omega = \Omega - \Omega_D \in \langle 5e-5 \div 1.25e-3 \rangle \Omega.$ Then we can perform modal analysis of undamped and unexcited system whose motion is described by equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}. \tag{5.4}$$

As experience has shown displacements of undamped systems vibrating by individual mode shapes are always in phase (0°) or in oposite phase (180°). It means that their time delays are equal to zero. Oposite phase can be expressed by oposite sign of amplitude. For this reason real aritmetics is sufficient for solution of (5.4) and it can be suppossed in form

$$\mathbf{q}(t) = \mathbf{v}\sin\Omega t\,, \tag{5.5}$$

where vibration in phase or in oposite phase is expressed by sign of vector component of **v**. Substituting (5.5) into (5.4) and canceling out by time function $\sin\Omega t$ we can come to

$$\left(\mathbf{K} - \Omega^2\mathbf{M}\right)\mathbf{v} = \mathbf{0}\,. \tag{5.6}$$

Equation (5.6) represents eigenvalue problem. Now we will supposse that defect of the matrix $\mathbf{K} - \Omega_i^2\mathbf{M}$ (difference between matrix order and its rank) will be equal to the multiplicity of eigenvalue Ω_i^2. Then we can say that the system is of simple structure. This assumption is used to be almost always satisfied. Many methods for numerical solution to the eigenvalue problem are presented in reference [5]. The non-trivial solution of (5.6) exists only if determinant of matrix $\mathbf{K} - \Omega^2\mathbf{M}$ is equal to zero (this matrix is singular) it means

$$\det\left(\mathbf{K} - \Omega^2\mathbf{M}\right) = 0. \tag{5.7}$$

Result of the solution of this characteristic equation is n real roots $\Omega_i^2,\ i = 1,2,......., n$. It is well known that eigenvalues of the problem $\mathbf{Ax} = \lambda\mathbf{Bx}$ are real when matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetrical. For vibrating systems the symbol Ω_i means so called eigenfrequencies [rad/s] - frequencies of system vibration without external effects-free vibration. Initial conditions decide about the contribution of the individual eigenfrequencies and mode shapes to the total response in free vibration. The equation (5.6) can be solved by standard methods for solution of eigenvalue matrix equation $\mathbf{Ax} = \lambda\mathbf{Bx}$. These methods are presented for instance in [5].
Substituting successively $\Omega_i^2,\ i = 1, 2, \dots, n$ into the equation

$$\left(\mathbf{K} - \Omega_i^2\mathbf{M}\right)\mathbf{v}_i = \mathbf{0} \tag{5.8}$$

we can obtain n mode shapes (eigenvectors) $\mathbf{v}_i,\ i = 1, 2, ..., n$. Because we také into account only systems of simple structure each one eigenvalue corresponds to one eigenvector. It means that for instance the eigenvalue Ω_i^2 whose multiplicity is n_i corresponds to

n_i eigenvectors. Eigenvectors corresponding to simple eigenvalues Ω_j^2 are determinated unequivocally except for multiplicative constant-matrix of coefficients in (5.8) has rank $n-1 \Rightarrow defect = 1$. It can be realized in such a way that the first component of the vector $\mathbf{v}_j$ is chosen and the other components are calculated to the end. For n_i multiple eigenvalue Ω_i^2 we choose n_i components for each individual eigenvector in such a way that chosen subvectors are mutually independent and then we complete the other components.

Orthogonality of eigenvectors

Let us write the equation (5.8) for *i*-th and *j*-th eigenvalue. The first and the second equation let us pre-multiply by $\mathbf{v}_j^T$ and $\mathbf{v}_i^T$, respectively and the second transpose. Then we can come to

$$\mathbf{v}_j^T\left(\mathbf{K}-\Omega_i^2\mathbf{M}\right)\mathbf{v}_i = 0\,, \tag{5.9}$$

$$\mathbf{v}_j^T\left(\mathbf{K}-\Omega_j^2\mathbf{M}\right)\mathbf{v}_i = 0\,. \tag{5.10}$$

Substracting this equations we have

$$\left(\Omega_i^2-\Omega_j^2\right)\mathbf{v}_j^T\mathbf{M}\mathbf{v}_i = 0. \tag{5.11}$$

The consequence of the last relation is

$$\mathbf{v}_j^T\mathbf{M}\mathbf{v}_i = \begin{cases} 0 \\ ? \end{cases} \text{ for } \begin{matrix} \Omega_i \neq \Omega_j \\ \Omega_i = \Omega_j \end{matrix}\;. \tag{5.12}$$

Respecting (5.11) and (5.12) we can say that eigenvectors are orthogonal for different eigenvalues. For multiple eigenvalues the eigenvectors need not be orthogonal and for this reason it is suitable to orthogonalize them. For the bellow presented reasons we would ask to be valid

$$\mathbf{v}_j^T\mathbf{M}\mathbf{v}_i = \delta_{ij}\,, \tag{5.13}$$

where δ_{ij} is Kronecker delta. Let us assemble eigenvectors satisfying (5.13) as columns of special matrix we can write

$$\mathbf{V} = \left[\mathbf{v}_1, \quad \mathbf{v}_2, \quad \cdots, \quad \mathbf{v}_n\right] \in \mathbf{R}^{n,n}\,. \tag{5.14}$$

The **V** matrix is called modal matrix satisfying condition

$$\mathbf{V}^T\mathbf{M}\mathbf{V} = \mathbf{I}, \tag{5.15}$$

where $\mathbf{I} \in \mathbf{R}^{n,n}$ is identity matrix. The relations (5.13) and (5.14) express not only the orthogonality but even condition of norm $\mathbf{v}_i^T\mathbf{M}\mathbf{v}_i = 1,\ i = 1, 2, \dots, n.$ For this reason it is suitable to perform an orthonormalization. Let us mark for this moment un-orthogonalized and un-normalized eigenvectors by tilda, it means

$$\tilde{\mathbf{V}} = \left[\tilde{\mathbf{v}}_1, \quad \tilde{\mathbf{v}}_2, \quad \cdots, \quad \tilde{\mathbf{v}}_n\right] \in \mathbf{R}^{n,n}. \tag{5.16}$$

This matrix satisfies condition

$$\tilde{\mathbf{V}}^T\mathbf{M}\tilde{\mathbf{V}} = \mathbf{Q}, \tag{5.17}$$

where $\mathbf{Q} \in \mathbf{R}^{n,n}$ is almost diagonal matrix having non-zero off-diagonal elements done by products of eigenvectors correponding of the multiple eigenvalue Ω_i^2. These submatrices of the n_i-th order lay on the diagonal of $\mathbf{Q}$. The orthonormalization can be performed in such a way that we decompose $\mathbf{Q}$ matrix by Choleskiôs decomposition into two triangular matrices

$$\mathbf{Q} = \mathbf{L}^T \mathbf{L}. \tag{5.18}$$

Matrix $\mathbf{Q}$ has to be symmetric and positively definite. This condition is satisfied when the $\mathbf{M}$ matrix is symmetric and positively definite too, it means decomposable by means of Choleski decomposition $\mathbf{M} = \mathbf{L}_1^T \mathbf{L}_1$. It can be proved that mass matrix $\mathbf{M}$ is always positively definite if it does not contain massless degrees of freedom. We can prove it expressing kinetic energy in quadratical form of general velocity vector (following form is valid for systems without rotation bodies).

$$E_k = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}.$$

The last relation holds always for $\dot{\mathbf{q}} \neq \mathbf{0}$, because kinetic energy can not have negative value. Then we can rewrite (5.17) into form

$$\tilde{\mathbf{V}}^T \mathbf{L}_1^T \mathbf{L}_1 \tilde{\mathbf{V}} = \mathbf{Q}. \tag{5.19}$$

From the relation (5.19) we can see that $\mathbf{x}^T \mathbf{Q} \mathbf{x} > 0$, for arbitrary vector $\mathbf{x}$ because following (5.19) it can be said that $\mathbf{x}^T \tilde{\mathbf{V}}^T \mathbf{L}_1^T \mathbf{L}_1 \tilde{\mathbf{V}} \mathbf{x} = \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} > 0$, where we marked $\tilde{\mathbf{x}} = \mathbf{L}_1 \tilde{\mathbf{V}} \mathbf{x}$. Let us mark $\mathbf{L} = \mathbf{L}_1 \tilde{\mathbf{V}}$ and we can obtain the relation (5.18) from (5.19). Substituting (5.18) into (5.17) and pre-multiplying whole equation by $\mathbf{L}^{-T}$ and post-multiplying by $\mathbf{L}^{-1}$ we can come to

$$\mathbf{L}^{-T} \tilde{\mathbf{V}}^T \mathbf{M} \tilde{\mathbf{V}} \mathbf{L}^{-1} = \mathbf{I}. \tag{5.20}$$

Comparing (5.20) and (5.15) we can say that orthonormalized modal matrix has form

$$\mathbf{V} = \tilde{\mathbf{V}} \mathbf{L}^{-1}. \tag{5.21}$$

This way of orthogonalization is called as Schmidtôs orthogonalization proces. The modal matrix obtained according to (5.21) satisfies requirement (5.13) and its compact form (5.15). For simple eigenvalues ($\Omega_i \neq \Omega_j \Leftrightarrow i \neq j$) the matrix $\mathbf{Q}$ is diagonal

$$\tilde{\mathbf{V}}^T \mathbf{M} \tilde{\mathbf{V}} = \mathbf{Q} = diag\left\{ \tilde{\mathbf{v}}_i^T \mathbf{M} \tilde{\mathbf{v}}_i \right\}, \tag{5.22}$$

and

$$\mathbf{L}^{-1} = diag\left\{ \frac{1}{\sqrt{\tilde{\mathbf{v}}_i^T \mathbf{M} \tilde{\mathbf{v}}_i}} \right\}. \tag{5.23}$$

It means that indicidual vectors are normalized following relation

$$\mathbf{v}_i = \frac{\tilde{\mathbf{v}}_i}{\sqrt{\tilde{\mathbf{v}}_i^T \mathbf{M} \tilde{\mathbf{v}}_i}}. \tag{5.24}$$

Respecting (5.13) let us rewrite (5.8) into form

$$\mathbf{v}_j^T\mathbf{K}\mathbf{v}_i = \Omega_i^2 \underbrace{\mathbf{v}_j^T\mathbf{M}\mathbf{v}_i}_{\delta_{ij}} = \Omega_i^2\delta_{ij}\,, \tag{5.25}$$

which can be further rewritten in compact form

$$\mathbf{V}^T\mathbf{K}\mathbf{V} = \text{ş}\ \ , \tag{5.26}$$

where $\text{ş} = diag\{\Omega_i^2\}$ is so called spectral matrix. The result of modal analysis of the mathematical models with weak damping and symmetrical metrices **M**, **B**, **K** is modal and spectral matrix, respectively, satisfying conditions

$$\boxed{\mathbf{V}^T\mathbf{M}\mathbf{V} = \mathbf{I},\ \ \mathbf{V}^T\mathbf{K}\mathbf{V} = \text{ş}}\,. \tag{5.27}$$

Relations (5.27) can be written inversely in the following way

$$\mathbf{M} = \mathbf{V}^{-T}\mathbf{V}^{-1} = \left(\mathbf{V}\mathbf{V}^T\right)^{-1}, \qquad \mathbf{K} = \mathbf{V}^{-T}\text{ş}\ \mathbf{V}^{-1}. \tag{5.28}$$

5.3 Free vibration of the weakly damped systems with commutative damping matrix

This sub-chapter will be focused on mathematical models with so called commutative damping matrix only. The commutative damping matrix is matrix satisfying condition

$$\mathbf{V}^T\mathbf{B}\mathbf{V} = \text{ũ} = diag\{2D_i\Omega_i\}, \tag{5.29}$$

which means that it is diagonalizable by means of modal matrix like matrices **M** and **K**. Because ũ is diagonal it has to commutate with any diagonal matrix e.g. ş . Equality

$$\text{ş}\ \text{ũ} = \text{ũ}\,\text{ş} \tag{5.30}$$

can be rewritten by means of relations (5.27) and (5.29) into form

$$\mathbf{V}^T\mathbf{K}\mathbf{V}\,\_\,\mathbf{V}^T\mathbf{B}\mathbf{V} = \mathbf{V}^T\mathbf{B}\mathbf{V}\,\_\,\mathbf{V}^T\mathbf{K}\mathbf{V}. \tag{5.31}$$

Inverting the first relation (5.27) ($\mathbf{V}^T\mathbf{M}\mathbf{V} = \mathbf{I}$) we can obtain

$$\mathbf{V}^{-1}\mathbf{M}^{-1}\mathbf{V}^{-1} = \mathbf{I}\,. \tag{5.32}$$

Let us insert this matrix into indicated blanks in (5.31) and obtain condition of damping matrix commutativity in form

$$\boxed{\mathbf{K}\mathbf{M}^{-1}\mathbf{B} = \mathbf{B}\mathbf{M}^{-1}\mathbf{K}\,.} \tag{5.33}$$

A special case of commutative damping matrix is matrix of so called proportional damping when the damping matrix is linear combination of mass and stiffness matrices, it means

$$\mathbf{B} = \alpha\mathbf{M} + \beta\mathbf{K}. \tag{5.34}$$

The possibility of diagonalize **B** matrix can be shown by substituting (5.34) into (5.29) and than we can come to relation

$$\mathbf{V}^T\mathbf{B}\mathbf{V} = \mathbf{V}^T(\alpha\,\mathbf{M} + \beta\,\mathbf{K})\mathbf{V} = \alpha\,\mathbf{V}^T\mathbf{M}\mathbf{V} + \beta\,\mathbf{V}^T\mathbf{K}\mathbf{V} = \alpha\,\mathbf{I} + \beta\boldsymbol{\S}\ . \tag{5.35}$$

The result is sum of two diagonal matrices which is diagonal matrix too. Let us remind the equation of motion describing free vibration of the systems with commutative damping matrix in form (5.1) with null right hand side

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{B}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \tag{5.36}$$

and null initial conditions

$$\mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0. \tag{5.37}$$

Instead of general displacement vector let us introduce vector of modal coordinates following form

$$\mathbf{q}(t) = \mathbf{V}\mathbf{x}(t).$$

Substituting the last relation into the equation of motion (5.36) and pre-multiplying whole equastion by matrix $\mathbf{V}^T$ we can come to matrix equation

$$\underbrace{\mathbf{V}^T\mathbf{M}\mathbf{V}}_{\mathbf{I}}\ddot{\mathbf{x}}(t) + \underbrace{\mathbf{V}^T\mathbf{B}\mathbf{V}}_{\tilde{\mathbf{u}}}\dot{\mathbf{x}}(t) + \underbrace{\mathbf{V}^T\mathbf{K}\mathbf{V}}_{\boldsymbol{\S}}\mathbf{x}(t) = \mathbf{0}, \tag{5.38}$$

which is system of *n* independent differential equations whose *i*-th one has form

$$\ddot{x}_i(t) + 2D_i\Omega_i\dot{x}_i(t) + \Omega_i^2 x_i(t) = 0 \tag{5.39}$$

and solution [7]

$$x_i(t) = e^{-D_i\Omega_i t}\left(\alpha_i \cos\Omega_{Di}t + \beta_i \sin\Omega_{Di}t\right), \tag{5.40}$$

where $\Omega_{Di} = \Omega_i\sqrt{1 - D_i^2}$ is *i*-th eigenfrequency of damping vibration. The relation (5.40) can be written down in matrix form

$$\mathbf{x}(t) = \mathbf{E}(t)[\mathbf{C}(t)\text{Ŭ} + \mathbf{S}(t)\text{б}], \tag{5.41}$$

$$\mathbf{E}(t) = \begin{bmatrix} e^{-D_1\Omega_1 t} & & & \\ & e^{-D_2\Omega_2 t} & & \\ & & \ddots & \\ & & & e^{-D_n\Omega_n t} \end{bmatrix}, \quad \mathbf{C}(t) = \begin{bmatrix} \cos\Omega_{D1}t & & & \\ & \cos\Omega_{D2}t & & \\ & & \ddots & \\ & & & \cos\Omega_{Dn}t \end{bmatrix}, \tag{5.42}$$

$$\mathbf{S}(t)=\begin{bmatrix}\sin\Omega_{D1}t & & & \\ & \sin\Omega_{D2}t & & \\ & & \ddots & \\ & & & \sin\Omega_{Dn}t\end{bmatrix},\quad \boldsymbol{\alpha}=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix},\quad \boldsymbol{\beta}=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{bmatrix}. \tag{5.43}$$

Turning back to the original generalized coordinates we can come to

$$\mathbf{q}(t)=\mathbf{V}\mathbf{x}(t)=\mathbf{V}\mathbf{E}(t)\left[\mathbf{C}(t)\boldsymbol{\alpha}+\mathbf{S}(t)\boldsymbol{\beta}\right]. \tag{5.44}$$

To obtain vectors of integral constants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we need the first derivation of general displacement vector (5.44) with respect to time having form

$$\dot{\mathbf{q}}(t)=-\mathbf{V}\mathbf{D}\boldsymbol{\Omega}\,\mathbf{E}(t)\left[\mathbf{C}(t)\boldsymbol{\alpha}+\mathbf{S}(t)\boldsymbol{\beta}\right]+\mathbf{V}\mathbf{E}(t)\boldsymbol{\Omega}_D\left[-\mathbf{S}(t)\boldsymbol{\alpha}+\mathbf{C}(t)\boldsymbol{\beta}\right], \tag{5.45}$$

where

$$\mathbf{D}=\begin{bmatrix}D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n\end{bmatrix},\quad \boldsymbol{\Omega}=\begin{bmatrix}\Omega_1 & & & \\ & \Omega_2 & & \\ & & \ddots & \\ & & & \Omega_n\end{bmatrix},\quad \boldsymbol{\Omega}_D=\begin{bmatrix}\Omega_{D1} & & & \\ & \Omega_{D2} & & \\ & & \ddots & \\ & & & \Omega_{Dn}\end{bmatrix}. \tag{5.46}$$

Substituting initial conditions (5.37) into relations (5.44) and (5.45) we have

$$\begin{aligned}&\mathbf{q}_0=\mathbf{V}\boldsymbol{\alpha}\Rightarrow\boldsymbol{\alpha}=\mathbf{V}^{-1}\mathbf{q}_0,\\ &\dot{\mathbf{q}}_0=-\mathbf{V}\mathbf{D}\boldsymbol{\Omega}\,\boldsymbol{\alpha}+\mathbf{V}\boldsymbol{\Omega}_D\boldsymbol{\beta}\Rightarrow\boldsymbol{\beta}=\boldsymbol{\Omega}_D^{-1}\mathbf{V}^{-1}\left(\dot{\mathbf{q}}_0+\mathbf{V}\mathbf{D}\boldsymbol{\Omega}\,\mathbf{V}^{-1}\mathbf{q}_0\right).\end{aligned} \tag{5.47}$$

Let us substitute the resulting vectors of integral constants into (5.44) and come to the relation

$$\mathbf{q}(t)=\mathbf{V}\mathbf{E}(t)\left[\mathbf{C}(t)\mathbf{V}^{-1}\mathbf{q}_0+\mathbf{S}(t)\boldsymbol{\Omega}_D^{-1}\mathbf{V}^{-1}\left(\dot{\mathbf{q}}_0+\mathbf{V}\mathbf{D}\boldsymbol{\Omega}\,\mathbf{V}^{-1}\mathbf{q}_0\right)\right]. \tag{5.48}$$

Vector of generalized velocities can be obtained by substitution of (5.47) into (5.45). Inversion of the diagonal matrix $\boldsymbol{\Omega}_D$ is very simple (all eigenfrequencies are non-zero) but inversion of matrix $\mathbf{V}$ can be simplified in such a way that the orthogonalization relation (5.15) will be post-multiplied by matrix $\mathbf{V}^{-1}$ and then we can write

$$\mathbf{V}^{-1}=\mathbf{V}^T\mathbf{M}. \tag{5.49}$$

Following the last relation we can rearrange (5.48) into form

$$\boxed{\mathbf{q}(t)=\mathbf{V}\mathbf{E}(t)\left[\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0+\mathbf{S}(t)\boldsymbol{\Omega}_D^{-1}\mathbf{V}^T\mathbf{M}\left(\dot{\mathbf{q}}_0+\mathbf{V}\mathbf{D}\boldsymbol{\Omega}\,\mathbf{V}^T\mathbf{M}\mathbf{q}_0\right)\right].} \tag{5.50}$$

For undamped systems these relations holds $D_i=0$, it means, that $\mathbf{E}(t)=\mathbf{I}$, $\mathbf{D}=\mathbf{0}$, $\boldsymbol{\Omega}_D=\boldsymbol{\Omega}$, $\mathbf{S}(t)=diag\{\sin\Omega_i t\}$, $\mathbf{C}(t)=diag\{\cos\Omega_i t\}$.

5.4 Forced vibration of the weakly damped systems with commutative damping matrix

For solution of forced vibration of these models we transfer equation of motion (5.1) into canonical form in modal coordinates in similar way as we done it in the case of free vibration (Eq. 5.38). Let us introduce transformation $\mathbf{q}(t)=\mathbf{V}\mathbf{x}(t)$ and substitute into (5.1) and then the resultant equation pre-multiply by $\mathbf{V}^T$. Now we can come to

$$\underbrace{\mathbf{V}^T\mathbf{M}\mathbf{V}}_{\mathbf{I}}\ddot{\mathbf{x}}(t)+\underbrace{\mathbf{V}^T\mathbf{B}\mathbf{V}}_{\text{ũ}}\dot{\mathbf{x}}(t)+\underbrace{\mathbf{V}^T\mathbf{K}\mathbf{V}}_{\text{ş}}\mathbf{x}(t)=\mathbf{V}^T\mathbf{f}(t)\,. \tag{5.51}$$

The last matrix equation corresponds again to the system of n independent differential equations

$$\ddot{x}_i(t)+2D_i\Omega_i\dot{x}_i(t)+\Omega_i^2x_i(t)=\mathbf{v}_i^T\mathbf{f}(t)\,. \tag{5.52}$$

To shorten writing in this moment we leave out subscript i and so we solve the equation

$$\ddot{x}(t)+2D\Omega\dot{x}(t)+\Omega^2x(t)=\mathbf{v}^T\mathbf{f}(t)\,, \tag{5.53}$$

whose solution consists of the homogeneous solution

$$x^H(t)=e^{-D\Omega t}(\alpha\cos\Omega_Dt+\beta\sin\Omega_Dt) \tag{5.54}$$

and particular solution which is formed by fundamental functions contained in the homogeneous solution (5.54). Both fundamental functions are marked by symbols

$$y_1(t)=e^{-D\Omega t}\cos\Omega_Dt,\qquad y_2(t)=e^{-D\Omega t}\sin\Omega_Dt. \tag{5.55}$$

It should be said that the fundamental functions (5.55) satisfy the equation (5.53) with zero right hand side. We will look for the particular solution of the equation (5.53) in the form of linear combination of fundamental functions (briefly written)

$$x^P(t)=c_1(t)y_1(t)+c_2(t)y_2(t)=c_1y_1+c_2y_2\,, \tag{5.56}$$

where $c_1(t)$ and $c_2(t)$ are desired time functions (varied constants). Let us substitute this supposed particular solution into the equation (5.53). For this reason let us differentiate the equation (5.56) with respect to time (briefly written)

$$\dot{x}^P(t)=\dot{c}_1y_1+c_1\dot{y}_1+\dot{c}_2y_2+c_2\dot{y}_2\,, \tag{5.57}$$

and choose from all solutions only solutions satisfying

$$\boxed{\dot{c}_1y_1+\dot{c}_2y_2=0}\,. \tag{5.58}$$

Then we obtain the first and second derivation of the particular solution in form

$$\dot{x}^P(t)=c_1\dot{y}_1+c_2\dot{y}_2\,, \tag{5.59}$$

$$\ddot{x}^P(t)=\dot{c}_1\dot{y}_1+c_1\ddot{y}_1+\dot{c}_2\dot{y}_2+c_2\ddot{y}_2\,. \tag{5.60}$$

Substituting (5.56), (5.59) and (5.60) into the original equation of motion and respecting fact that the nfundamental solution satisfies corresponding homogeneous equation we can write (expressions marked by star and double star are in summ equal to zero)

$$\dot{c}_1\dot{y}_1+\underbrace{c_1\ddot{y}_1}_{*}+\dot{c}_2\dot{y}_2+\underbrace{c_2\ddot{y}_2}_{**}+2D\Omega\left(\underbrace{c_1\dot{y}_1}_{*}+\underbrace{c_2\dot{y}_2}_{**}\right)+\Omega^2\left(\underbrace{c_1y_1}_{*}+\underbrace{c_2y_2}_{**}\right)=\mathbf{v}^T\mathbf{f}(t) \tag{5.61}$$

and then

$$\boxed{\dot{c}_1\dot{y}_1 + \dot{c}_2\dot{y}_2 = \mathbf{v}^T\mathbf{f}(t)}. \tag{5.62}$$

From Ăframedñ equations (5.58) and (5.62) we can get

$$\dot{c}_1 = \frac{-\mathbf{v}^T\mathbf{f}(t)y_2}{y_1\dot{y}_2 - \dot{y}_1 y_2}, \quad \dot{c}_2 = \frac{\mathbf{v}^T\mathbf{f}(t)y_1}{y_1\dot{y}_2 - \dot{y}_1 y_2}. \tag{5.63}$$

Respecting (5.55) denominators of time functions can be expressed in form

$$y_1\dot{y}_2 - \dot{y}_1 y_2 = \Omega_D e^{-2D\Omega t} \tag{5.64}$$

and after substitution into (5.63) we can come to

$$\dot{c}_1 = -\frac{1}{\Omega_D}\mathbf{v}^T\mathbf{f}(t)e^{D\Omega t}\sin\Omega_D t = \frac{dc_1}{dt}, \qquad \dot{c}_2 = \frac{1}{\Omega_D}\mathbf{v}^T\mathbf{f}(t)e^{D\Omega t}\cos\Omega_D t = \frac{dc_2}{dt}. \tag{5.65}$$

Having multiplied both equations by dt we separate unknowns and after integration obtain

$$c_1 = -\frac{1}{\Omega_D}\int_0^t \mathbf{v}^T\mathbf{f}(t)e^{D\Omega\tau}\sin\Omega_D\tau\, d\tau, \qquad c_2 = \frac{1}{\Omega_D}\int_0^t \mathbf{v}^T\mathbf{f}(t)e^{D\Omega\tau}\cos\Omega_D\tau\, d\tau. \tag{5.66}$$

Substituting (5.66) into (5.56) we can write

$$x^P(t) = -\frac{1}{\Omega_D}\int_0^t \mathbf{v}^T\mathbf{f}(t)e^{D\Omega\tau}\sin\Omega_D\tau\, d\tau\; e^{-D\Omega t}\cos\Omega_D t +$$

$$+\frac{1}{\Omega_D}\int_0^t \mathbf{v}^T\mathbf{f}(t)e^{D\Omega\tau}\cos\Omega_D\tau\, d\tau\; e^{-D\Omega t}\sin\Omega_D t = \frac{1}{\Omega_D}\int_0^t \mathbf{v}^T\mathbf{f}(t)e^{-D\Omega(t-\tau)}\sin\Omega_D(t-\tau)\, d\tau. \tag{5.67}$$

Let us turn back to the subscripts i for individual solutions. Total solution of i-th modal coordinate can be expressed in form of summ of homogeneous (5.54) and paticular (5.67) solutions (written with subscript i)

$$x_i(t) = e^{-D_i\Omega_i t}(\alpha_i\cos\Omega_{Di}t + \beta_i\sin\Omega_{Di}t) + \frac{1}{\Omega_{Di}}\int_0^t \mathbf{v}_i^T\mathbf{f}(\tau)e^{-D_i\Omega_i(t-\tau)}\sin\Omega_{Di}(t-\tau)\, d\tau. \tag{5.68}$$

The last relation can be written in compact form

$$\mathbf{x}(t) = \mathbf{E}(t)[\mathbf{C}(t)\check{\mathbf{U}} + \mathbf{S}(t)\text{ɓ}] + \mathbf{q}_D^{-1}\int_0^t \mathbf{E}(t-\tau)\mathbf{S}(t-\tau)\mathbf{V}^T\mathbf{f}(\tau)d\tau\,. \tag{5.69}$$

Further we can turn back to generalized coordinates by means of transformation relation $\mathbf{q}(t) = \mathbf{V}\mathbf{x}(t)$, and then we obtain

$$\mathbf{q}(t) = \mathbf{V}\mathbf{E}(t)[\mathbf{C}(t)\check{\mathbf{U}} + \mathbf{S}(t)\text{ɓ}] + \mathbf{V}\mathbf{q}_D^{-1}\int_0^t \mathbf{E}(t-\tau)\mathbf{S}(t-\tau)\mathbf{V}^T\mathbf{f}(\tau)d\tau. \tag{5.70}$$

Matrices contained in (5.70) were defined in relations (5.42) until (5.46). Vectors of integral constants $\check{\mathbf{U}}$ and ɓ can be obtained from initial conditions. Therefore we have to derivative last relation with respect to time. Time is contained not only in integrand but in limit of integration, too. For this reason let us remind the relation for derivative of integral

$$g(t) = \int_{\varphi_1(t)}^{\varphi_2(t)} f(t,\tau)d\tau \tag{5.71}$$

with respect to parameter t which has form

$$\frac{dg}{dt}=\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\partial f(t,\tau)}{\partial t}d\tau+\varphi_2^{\backslash}(t)f[\varphi_2(t),t]-\varphi_1^{\backslash}(t)f[\varphi_1(t),t].\tag{5.72}$$

In this case

$$\frac{d}{dt}\left\{\int_0^t\mathbf{E}(t-\tau)\mathbf{S}(t-\tau)\mathbf{V}^T\mathbf{f}(\tau)d\tau\right\}=\int_0^t[-\mathbf{Dq}\;\mathbf{E}(t-\tau)\mathbf{S}(t-\tau)+\mathbf{q}_D\mathbf{E}(t-\tau)\mathbf{C}(t-\tau)]\mathbf{V}^T\mathbf{f}(\tau)d\tau.\tag{5.73}$$

The second and third term in (5.72) are equal to zero. Now we can express differentiation of the whole vector of generalized displacements (5.70) in form

$$\dot{\mathbf{q}}(t)=-\mathbf{VDq}\;\mathbf{E}(t)[\mathbf{C}(t)\breve{\mathbf{U}}+\mathbf{S}(t)\mathbf{б}]+\mathbf{VE}(t)\mathbf{q}_D[\mathbf{C}(t)\mathbf{б}-\mathbf{S}(t)\breve{\mathbf{U}}]+$$
$$+\mathbf{Vq}_D^{-1}\int_0^t[-\mathbf{Dq}\;\mathbf{E}(t-\tau)\mathbf{S}(t-\tau)+\mathbf{q}_D\mathbf{E}(t-\tau)\mathbf{C}(t-\tau)]\mathbf{V}^T\mathbf{f}(\tau)d\tau.\tag{5.74}$$

Substituting initial conditions into (5.70) and (5.74) we can write ($t=0$)

$$\mathbf{q}_0=\mathbf{V}\breve{\mathbf{U}}\;\Rightarrow\;\breve{\mathbf{U}}=\mathbf{V}^{-1}\mathbf{q}_0=\mathbf{V}^T\mathbf{Mq}_0,\tag{5.75}$$

$$\dot{\mathbf{q}}_0=-\mathbf{VDq}\;\underbrace{\mathbf{V}^T\mathbf{Mq}_0}_{\breve{\mathbf{U}}}+\mathbf{Vq}_D\mathbf{б}\;\Rightarrow\mathbf{б}=\mathbf{q}_D^{-1}\left(\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0+\mathbf{Dq}\;\mathbf{V}^T\mathbf{Mq}_0\right).\tag{5.76}$$

Total solution then can be written as

$$\mathbf{q}(t)=\mathbf{VE}(t)\left[\mathbf{C}(t)\mathbf{V}^T\mathbf{Mq}_0+\mathbf{S}(t)\mathbf{q}_D^{-1}\left(\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0+\mathbf{Dq}\;\mathbf{V}^T\mathbf{Mq}_0\right)\right]+\mathbf{Vq}_D^{-1}\int_0^t\mathbf{E}(t-\tau)\mathbf{S}(t-\tau)\mathbf{V}^T\mathbf{f}(\tau)d\tau.\tag{5.77}$$

As it is well known from the theory of differential equations particular solution can be arbitrary function satisfying (besides desired conditions of differentiability) corresponding differential equation with right hand side. In this case when we came to particular solution by means of variation of constants (convolutory integral (5.67)) this solution corresponds to total solution of equation with zero initial conditions. For this result integral constants of the total solution are identical with those ones of itself homogeneous solution. Therefore the first summand in (5.77) is identical with solution of free vibration (5.50). We should aware that $\mathbf{V}^{-1}=\mathbf{V}^T\mathbf{M}$. Now it can be concluded that total solution (5.77) consists of solution of equation (5.1) without excitation and the solution with excitation but with null initial conditions. In case when we choose or estimate particular solution in the other form than convolutory integral (5.67), e.g. in form of excitation, it is necessary to compute integral constants from the total solution.

5.5 Particular solution estimation of the harmonically excited system using complex form of the equation of motion

Let us suposse that behaviour of the vibrating system is described by equation of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{B}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}_c\cos\omega t+\mathbf{f}_s\sin\omega t\,, \tag{5.78}$$

where $\mathbf{f}_c$ and $\mathbf{f}_s$ is vector of sine and cosine amplitudes, respectively. Both vectors are real valued. The right hand side can be written in form

$$\mathbf{f}_c\cos\omega t+\mathbf{f}_s\sin\omega t=\mathbf{f}_c\cos\omega t+\mathbf{f}_s\cos\left(\omega t-\frac{\pi}{2}\right). \tag{5.79}$$

The real excitation (5.79) can be seen as real component of complex excitation

$$\mathbf{f}_c\cos\omega t+\mathbf{f}_s\cos\left(\omega t-\frac{\pi}{2}\right)=\mathbf{f}_c e^{i\omega t}+\mathbf{f}_s e^{i\left(\omega t-\frac{\pi}{2}\right)}=\mathbf{f}_c e^{i\omega t}+\mathbf{f}_s e^{i\omega t}\underbrace{e^{-i\frac{\pi}{2}}}_{-i}. \tag{5.80}$$

Introducing complex amplitude of the excitation the last relation can be rewritten into form

$$\left(\mathbf{f}_c-i\mathbf{f}_s\right)e^{i\omega t}=\breve{\mathbf{f}}e^{i\omega t}, \tag{5.81}$$

where $\breve{\mathbf{f}}$ is mentioned vector of complex amplitudes of excitation. Verbally we can relation (5.81) comment in such a way, that the real part of complex amplitude vector of harmonic excitation is formed by cosine amplitude vector while the imaginary part is formed by negative taken sine amplitude vector. Substituting relation for complex excitation (5.81) into the equation of motion (5.78) and introducing vector of complex displacements $\breve{\mathbf{q}}(t)$ in the similar way like complex excitation we can write

$$\mathbf{M}\ddot{\breve{\mathbf{q}}}(t)+\mathbf{B}\dot{\breve{\mathbf{q}}}(t)+\mathbf{K}\breve{\mathbf{q}}(t)=\breve{\mathbf{f}}e^{i\omega t}. \tag{5.82}$$

Let us estimate particular solution according to the right hand side in form $\breve{\mathbf{q}}(t)=\mathbf{q}_a e^{i\omega t}$ and substitute it into the last equation. Than we obtain complex amplitude form of the equation (5.82)

$$\left(-\omega^2\mathbf{M}+i\omega\mathbf{B}+\mathbf{K}\right)\mathbf{q}_a=\breve{\mathbf{f}}, \tag{5.83}$$

from which we obtain the complex amplitude vector of displacements in form

$$\mathbf{q}_a=\left(-\omega^2\mathbf{M}+i\omega\mathbf{B}+\mathbf{K}\right)^{-1}\breve{\mathbf{f}}=\mathbf{Z}^{-1}(\omega)\breve{\mathbf{f}}=\mathbf{H}(\omega)\breve{\mathbf{f}}, \tag{5.84}$$

where $\mathbf{Z}(\omega)\in\mathbf{C}^{n,n}$ is matrix of dynamic stiffness and $\mathbf{H}(\omega=2\pi f)\in\mathbf{C}^{n,n}$ is matrix of dynamic compliance (matrix of frequency transmission). Real particular solution is than real part of the complex vector of displacements

$$\mathrm{Re}\left\{\mathbf{q}_a e^{i\omega t}\right\}=\mathrm{Re}\left\{\left(\mathbf{q}_r+i\mathbf{q}_i\right)\left(\cos\omega t+i\sin\omega t\right)\right\}=\mathbf{q}_r\cos\omega t-\mathbf{q}_i\sin\omega t. \tag{5.85}$$

Example 5.1

Let us show for ilustration the total solution of the harmonically excited undamped system having *n* DOF (degrees of freedom) by means of estimated particular solution and by means of particular solution in form of convolutory integral. Let us suposse that behaviour of the vibrating system is described by equation of motion and initial conditions

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}_c\cos\omega t+\mathbf{f}_s\sin\omega t, \quad \mathbf{q}(0)=\mathbf{q}_0,\ \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_0. \tag{5.86}$$

Respecting (5.81) we rewrite the last equation into form

$$\mathbf{M}\ddot{\hat{\mathbf{q}}}(t)+\mathbf{K}\hat{\mathbf{q}}(t)=\hat{\mathbf{f}}e^{i\omega t}. \tag{5.87}$$

Having performed modal analysis we obtain modal matrix $\mathbf{V}$ and spectral matrix $\boldsymbol{\Lambda}$ from matrices $\mathbf{M}$ and $\mathbf{K}$. By means of these matrices we can write down homogeneous solution which has in both cases the same form (5.44)

$$\hat{\mathbf{q}}^H(t)=\mathbf{V}\mathbf{x}^H(t)=\mathbf{V}[\mathbf{C}(t)\hat{\mathbf{x}}_0+\mathbf{S}(t)\dot{\hat{\mathbf{x}}}_0]. \tag{5.88}$$

All symbols in (5.88) were explained above. Superscript H notes in this case homogeneous solution. The cap means that solution can be complex valued. Vectors of integral constants $\hat{\mathbf{x}}_0$ and $\dot{\hat{\mathbf{x}}}_0$ will be computed differently in following two cases. Respecting fact, that the system is undamped, the matrix $\mathbf{E}(t)$ takes form

$$\mathbf{E}(t)=\mathbf{I}. \tag{5.89}$$

a) Total solution obtained by means of particular solution estimation and modal transformation

As a starting point we can perform modal transformation

$$\hat{\mathbf{q}}^P(t)=\mathbf{V}\mathbf{x}^P(t). \tag{5.90}$$

After substitution (5.90) into (5.87) and pre-multiplying by $\mathbf{V}^T$ we can respecting (5.27) come to

$$-\omega^2\ddot{\mathbf{x}}^P(t)+\boldsymbol{\Lambda}\mathbf{x}^P(t)=\mathbf{V}^T\hat{\mathbf{f}}e^{i\omega t}. \tag{5.91}$$

Particular solution of the equation (5.91) can be estimated according to the right hand side in form

$$\mathbf{x}^P(t)=\mathbf{x}_a e^{i\omega t}. \tag{5.92}$$

Substituting the last relation into (5.91) we can come to the amplitude equation

$$\left(\boldsymbol{\Lambda}-\omega^2\mathbf{I}\right)\mathbf{x}_a=\mathbf{V}^T\hat{\mathbf{f}}, \tag{5.93}$$

which has solution

$$\mathbf{x}_a=\left(\boldsymbol{\Lambda}-\omega^2\mathbf{I}\right)^{-1}\mathbf{V}^T\hat{\mathbf{f}}. \tag{5.94}$$

After substitution (5.94) into (5.92) we can come to relation

$$\mathbf{x}^P(t)=\left(\boldsymbol{\Lambda}-\omega^2\mathbf{I}\right)^{-1}\mathbf{V}^T\hat{\mathbf{f}}e^{i\omega t}. \tag{5.95}$$

Turning back to original coordinates following (5.90) we have

$$\hat{\mathbf{q}}^P(t)=\mathbf{V}\left(\boldsymbol{\Lambda}-\omega^2\mathbf{I}\right)^{-1}\mathbf{V}^T\hat{\mathbf{f}}e^{i\omega t}. \tag{5.96}$$

The total solution is summ of the homogeneous (5.88) and particular (5.96) solution and has form

$$\hat{\mathbf{q}}(t) = \mathbf{V}[\mathbf{C}(t)\hat{\boldsymbol{\alpha}} + \mathbf{S}(t)\hat{\boldsymbol{\beta}}] + \mathbf{V}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}e^{i\omega t}. \tag{5.97}$$

The real solution corresponds to real component of the complex solution (5.97). Homogeneous solution (5.88) and thus the first summand in (5.97) is real therefore homogeneous solution was not marked by cap. (Modal and spectral matrices and eigenfrequencies are real for this investigated system). For now we stay at total complex solution. Transfer to the real solution will be performed after computation of integral constants. They can be determinated from initial conditions. For this reason let us derivative (5.97) and write

$$\dot{\hat{\mathbf{q}}}(t) = \mathbf{V}[-\boldsymbol{\Omega}\mathbf{S}(t)\hat{\boldsymbol{\alpha}} + \boldsymbol{\Omega}\mathbf{C}(t)\hat{\boldsymbol{\beta}}] + i\omega\mathbf{V}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}e^{i\omega t}. \tag{5.98}$$

After substitution of initial condition $\mathbf{q}(0) = \mathbf{q}_0$ into (5.97) we come to

$$\mathbf{q}_0 = \mathbf{V}\hat{\boldsymbol{\alpha}} + \mathbf{V}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}, \tag{5.99}$$

and from (5.99) ($\mathbf{V}^{-1} = \mathbf{V}^T\mathbf{M}$)

$$\hat{\boldsymbol{\alpha}} = \mathbf{V}^T\mathbf{M}\mathbf{q}_0 - (\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}. \tag{5.100}$$

Now let us substitute initial condition $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$ into (5.98) and have

$$\dot{\mathbf{q}}_0 = \mathbf{V}\boldsymbol{\Omega}\hat{\boldsymbol{\beta}} + i\omega\mathbf{V}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}, \tag{5.101}$$

and from (5.101) ($\mathbf{V}^{-1} = \mathbf{V}^T\mathbf{M}$)

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\Omega}^{-1}\mathbf{V}^T\mathbf{M}\left[\dot{\mathbf{q}}_0 - i\omega\mathbf{V}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}\right]. \tag{5.102}$$

From vector form of integral constants follows that they are influenced by excitation too (by right hand side of the equation of motion (5.87)) and are generally complex valued. This rule does not hold only in case when particular solution is computed by means of variation of constants whose result is convolutory integral. Now let us express total solution (5.97) with substituted vectors of integral constants (5.100) and (5.102). After small rearrangement we can come to

$$\begin{aligned}\hat{\mathbf{q}}(t) = {} & \mathbf{V}\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 - \mathbf{V}\mathbf{C}(t)(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}} + \mathbf{V}\mathbf{S}(t)\boldsymbol{\Omega}^{-1}\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0 - \\ & - i\omega\mathbf{V}\mathbf{S}(t)\boldsymbol{\Omega}^{-1}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}} + \mathbf{V}(\boldsymbol{\Lambda} - \omega^2\mathbf{I})^{-1}\mathbf{V}^T\hat{\mathbf{f}}e^{i\omega t}.\end{aligned} \tag{5.103}$$

The real solution corresponds to the real part of the last relation. It will be shown below.

b) Total solution by means of particular solution in the form of convolutory integral and modal transformation

In this case we can use homogeneous solution directly in form (5.50) where $\mathbf{E}(t) = \mathbf{I}, \quad \boldsymbol{\Omega}_D \equiv \boldsymbol{\Omega}, \ \mathbf{D} = \mathbf{0}.$ Than we have

$$\mathbf{q}^H(t) = \mathbf{V}\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 + \mathbf{V}\mathbf{S}(t)\boldsymbol{\Omega}_D^{-1}\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0. \tag{5.104}$$

Now the homogeneous solution was marked without cap because it is real valued. Particular solution has form of convolutory integral (the second summand in (5.70)) where

$$\mathbf{E}(t-\tau) = \mathbf{I}, \quad \boldsymbol{\Omega}_D \equiv \boldsymbol{\Omega}, \ \mathbf{f}(\tau) = \hat{\mathbf{f}}e^{i\omega\tau}.$$

It takes a form

$$\breve{\mathbf{q}}^P(t) = \mathbf{V}\mathbf{\Omega}_D^{-1}\int_0^t \mathbf{S}(t-\tau)\mathbf{V}^T\breve{\mathbf{f}}e^{i\omega\tau}d\tau. \tag{5.105}$$

Particular solution is again marked by cap because of complex excitation. Repscting

$$\int_0^t e^{i\omega\tau}\sin\Omega_\nu(t-\tau)d\tau = \frac{1}{\Omega_\nu^2-\omega^2}\left(\Omega_\nu e^{i\omega t} - i\omega\sin\Omega_\nu t - \Omega_\nu\cos\Omega_\nu t\right), \tag{5.106}$$

we can rewrite (5.105) into form

$$\breve{\mathbf{q}}^P(t) = \mathbf{V}\left(\mathbf{\Lambda} - \omega^2\mathbf{I}\right)^{-1}\left[\mathbf{I}e^{i\omega t} - i\omega\mathbf{\Omega}_D^{-1}\mathbf{S}(t) - \mathbf{C}(t)\right]\mathbf{V}^T\breve{\mathbf{f}} \tag{5.107}$$

and total solution is sum of (5.104) and (5.107)

$$\breve{\mathbf{q}}(t) = \mathbf{V}\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 + \mathbf{V}\mathbf{S}(t)\mathbf{\Omega}_D^{-1}\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{V}\left(\mathbf{\Lambda} - \omega^2\mathbf{I}\right)^{-1}\left[\mathbf{I}e^{i\omega t} - i\omega\mathbf{\Omega}_D^{-1}\mathbf{S}(t) - \mathbf{C}(t)\right]\mathbf{V}^T\breve{\mathbf{f}}; \tag{5.108}$$

which is identical form with (5.103). In this second case the homogeneous solution is formed by the first two terms of (5.108). It is as mentioned above real valued. Now we can express real solution (solution describing real behavior of the system) as real part of complex solution (5.108). Let us take into account that

$$\breve{\mathbf{f}} = \mathbf{f}_c - i\mathbf{f}_s \quad \text{a} \quad e^{i\omega t} = \cos\omega t + i\sin\omega t, \tag{5.109}$$

and so

$$e^{i\omega t}\breve{\mathbf{f}} = \mathbf{f}_c\cos\omega t + \mathbf{f}_s\sin\omega t + i(\mathbf{f}_c\sin\omega t - \mathbf{f}_s\cos\omega t) \Rightarrow \mathrm{Re}\left\{e^{i\omega t}\breve{\mathbf{f}}\right\} = \mathbf{f}_c\cos\omega t + \mathbf{f}_s\sin\omega t, \tag{5.110}$$

$$i\omega\breve{\mathbf{f}} = i\omega\mathbf{f}_c + \omega\mathbf{f}_s \Rightarrow \mathrm{Re}\left\{i\omega\breve{\mathbf{f}}\right\} = \omega\mathbf{f}_s, \tag{5.111}$$

$$\breve{\mathbf{f}} = \mathbf{f}_c - i\mathbf{f}_s \Rightarrow \mathrm{Re}\left\{\breve{\mathbf{f}}\right\} = \mathbf{f}_c. \tag{5.112}$$

Following the last three relations we can write real part of the equation (5.108) in form

$$\boxed{\begin{aligned}&\mathbf{q}(t) = \mathbf{V}\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 + \mathbf{V}\mathbf{S}(t)\mathbf{\Omega}_D^{-1}\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0 + \\ &+ \mathbf{V}\left(\mathbf{\Lambda} - \omega^2\mathbf{I}\right)^{-1}\left[\mathbf{V}^T(\mathbf{f}_c\cos\omega t + \mathbf{f}_s\sin\omega t) - \omega\mathbf{\Omega}_D^{-1}\mathbf{S}(t)\mathbf{V}^T\mathbf{f}_s - \mathbf{C}(t)\mathbf{V}^T\mathbf{f}_c\right].\end{aligned}} \tag{5.113}$$

Example 5.2

As an example let us present calculation of system response to the impulse excitation in two ways. The first presents the change of initial velocities caused by force impulse and the next computation corresponds to the free vibration with changed initial conditions. Let suposse that the equation of motion including initial conditions has form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{B}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{i}\delta(t), \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0, \tag{5.114}$$

where all symbols on the left hand side were explained above. The right hand side is formed by vector **i** multiplied by unit Dirac impulse. The vector **i** can express distribution of the impulse over the system. Further we suposse that the impulse takes for infinitely short time.

a) solution of free vibration by means of change of initial conditions

Respecting the law of change of momentum we can compute a new vector of initial velocities

$$\mathbf{M}\dot{\bar{\mathbf{q}}}_0 - \mathbf{M}\dot{\mathbf{q}}_0 = \int \mathbf{i}\delta(t)dt = \mathbf{i}\underbrace{\int \delta(t)dt}_{1} = \mathbf{i} \;\;\Rightarrow\;\; \dot{\bar{\mathbf{q}}}_0 = \dot{\mathbf{q}}_0 + \mathbf{M}^{-1}\mathbf{i}. \tag{5.115}$$

The vector of initial displacements stays the same as the original one. Now we can solve the equation for free vibration of the system whose behaviour is described by differential equation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{B}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\bar{\mathbf{q}}}_0 = \dot{\mathbf{q}}_0 + \mathbf{M}^{-1}\mathbf{i}\,. \tag{5.116}$$

Substituting $\dot{\bar{\mathbf{q}}}_0$ into (5.50) instead of $\dot{\mathbf{q}}_0$ we can come to

$$\mathbf{q}(t) = \mathbf{V}\mathbf{E}(t)\left[\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 + \mathbf{S}(t)\mathbf{q}_D^{-1}\left(\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{V}^T\mathbf{i} + \mathbf{D}\mathbf{q}\ \mathbf{V}^T\mathbf{M}\mathbf{q}_0\right)\right]. \tag{5.117}$$

b) solution to forced vibration

Replacing $\mathbf{f}(\tau)$ in (5.77) by $\mathbf{i}\delta(\tau)$ we can write

$$\mathbf{q}(t) = \mathbf{V}\mathbf{E}(t)\left[\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 + \mathbf{S}(t)\mathbf{q}_D^{-1}\left(\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{D}\mathbf{q}\ \mathbf{V}^T\mathbf{M}\mathbf{q}_0\right)\right] + \mathbf{V}\mathbf{q}_D^{-1}\int_0^t \mathbf{E}(t-\tau)\mathbf{S}(t-\tau)\mathbf{V}^T\mathbf{i}\delta(\tau)d\tau. \tag{5.118}$$

Let us use a property of Dirac impulse

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0). \tag{5.119}$$

Than integral in (5.118) will be equal to functional value of integrand in position $\tau = 0$ which can be written down as

$$\mathbf{V}\mathbf{q}_D^{-1}\int_0^t \mathbf{E}(t-\tau)\mathbf{S}(t-\tau)\mathbf{V}^T\mathbf{i}\delta(\tau)d\tau = \mathbf{V}\mathbf{q}_D^{-1}\mathbf{E}(t)\mathbf{S}(t)\mathbf{V}^T\mathbf{i} = \mathbf{V}\mathbf{E}(t)\mathbf{S}(t)\mathbf{q}_D^{-1}\mathbf{V}^T\mathbf{i}, \tag{5.120}$$

because diagonal matrices $\mathbf{E}(t)$, $\mathbf{S}(t)$, $\mathbf{q}_D^{-1}$ commutate whole expression (5.118) takes a form

$$\mathbf{q}(t) = \mathbf{V}\mathbf{E}(t)\left[\mathbf{C}(t)\mathbf{V}^T\mathbf{M}\mathbf{q}_0 + \mathbf{S}(t)\mathbf{q}_D^{-1}\left(\mathbf{V}^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{V}^T\mathbf{i} + \mathbf{D}\mathbf{q}\ \mathbf{V}^T\mathbf{M}\mathbf{q}_0\right)\right], \tag{5.121}$$

which is expression identical with (5.117)

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